

Abstract

In this paper, we investigate the rise of elementary symmetric polynomials in compositions under the hyperbolic tangent function. We first split the composition into two cases, where we examine the structure when an odd number and an even number of elements are composed. Through the use of floor functions, we unify the two cases, resulting in a closed form generalization. We prove this structure using mathematical induction.

Our contribution lies in applying this structure to special relativity, where we convert rapidities into velocities. The structure is valid for only collinear velocity compositions in 1 dimension. We then further prove that the structure obeys the postulates of special relativity by showing that it does not exceed the speed of light.

In addition, we develop a generalized formula for the total effect of the Lorentz factor for any number of collinear or parallel boosts. A boost is a transformation of coordinates between two inertial coordinate systems moving at constant relative velocity to one another without rotation.

In a purely theoretical scenario, we implement the structure to analyze the sufficient conditions by which the Lorentz factor converges for an infinite number of velocity additions. Finally, we derive generalizations of the Lorentz transformations.

Introduction

Special Relativity provides Einstein’s nonlinear velocity addition formula for two collinear velocities. However, the structure becomes cumbersome after more than two additions.

Could there be an underlying pattern to this increasingly complex structure?

This project’s aim is to derive a closed form formula of the resultant velocity instead of repeatedly applying Einstein’s formula. The following equation is velocity addition for two collinear velocities, where (c) is the speed of light:

$$v_1 \oplus v_2 = \frac{v_1 + v_2}{1 + \frac{(v_1)(v_2)}{c^2}}$$

Since the addition formula is nonlinear, it becomes highly complex of multiple additions.

Methods

We use rapidity, which is a parameter related to velocity through the hyperbolic tangent function, to simplify our approach. Rapidity is particularly useful because it adds linearly:

$$v_1 \oplus v_2 \oplus \dots \oplus v_i = c \times \tanh\left(\sum_{j=1}^i \varphi_j\right)$$

Secondly, we use elementary symmetric polynomials (ESPs) to describe the combinatorial structure that emerges after repeated velocity additions. They are defined as

$$e_0 = 1, \quad e_r(x_1, \dots, x_i) = \sum_{1 \leq j_1 < \dots < j_r \leq i} x_{j_1} \cdots x_{j_r}.$$

In other words, the r-th elementary symmetric polynomial is the sum of all products of r distinct variables chosen from the set{ $x_1, x_2, \dots, x_i$ }.

Results

The pattern that emerges can be seen from small cases:
When i=2

$$\tanh(\varphi_1 + \varphi_2) = \frac{e_1}{e_0 + e_2}$$

When i = 3

$$\tanh(\varphi_1 + \varphi_2 + \varphi_3) = \frac{e_1 + e_3}{e_0 + e_2}$$

When i =4

$$\tanh(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = \frac{e_1 + e_3}{e_0 + e_2 + e_4}$$

As we can see, the numerator consists of all odd-indexed ESPs, and the denominator consists of all even-indexed ESPs.

$$\tanh\left(\sum_{j=1}^i \varphi_j\right) =$$

$$= \frac{e_1 + e_3 + e_5 + \dots}{e_0 + e_2 + e_4 + \dots}$$

$$\frac{\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} e_{2r+1}}{\sum_{r=0}^{\lfloor i/2 \rfloor} e_{2r}}$$

$$v_1 \oplus v_2 \oplus \dots \oplus v_i$$

$$= \frac{\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} (1/c^2) e_{2r+1}}{\sum_{r=0}^{\lfloor i/2 \rfloor} (1/c^2) e_{2r}}$$

Discussion

As we have demonstrated, the structure is combinatorial and can be describe precisely by ESPs. The coefficient, $1/c^2$, emerges when converting from rapidities to velocities, ensuring that the resultant velocity is below the speed of light, in accordance to Special Relativity.

The structure, however, is only valid for one-dimensional motion between inertial reference frames moving at a constant velocity to one another.

This provides a foundation to generalize other mathematical objects in Special Relativity. For example, the Lorentz factor can be shown to exhibit a combinatorial structure since it depends on velocity.

Conclusion

We have demonstrated an underlying pattern that emerges from increasing additions of relativistic velocity, which can be described by ESPs.

By first investigating rapidities and their relationship with velocity through the hyperbolic tangent, we established the closed-form formula for the resultant velocity after an arbitrary additions.

Extending this framework to three-dimensional motion and to relativistic kinetic energy presents promising future research.

References

[1] Einstein, A. (1905). On the Electrodynamics of Moving Bodies.
[2] Holland, N. (2026). The Combinatorial Structure of Special Relativity. Oxford Journal of Student Scholarship, 7(1).