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Calculating the Escape Velocity and Lower Mass Bound of a Body within a Dark Matter Halo

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ABSTRACT

Dark matter remains to be one of the most speculative and mysterious substances from our current perspective in astrophysics. This paper will aim to provide educated hypotheses on the fermionic particle which makes up dark matter, by calculating the lower bound of the particle mass in an average dwarf spheroidal galaxy. This bound can be obtained through deriving the Escape Velocity based on calculations regarding Yukawa forces between dark matter particles. This can be done using the Pauli Exclusion Principle regarding fermionic particles to determine the mass density of the galaxy. This mass density can be calculated using models of the Jeans Analysis and Stellar Kinematics employing the Poisson Equation and Newton's Shell Theorems.

Solving dark matter can answer some of our universe's most important questions. Index Terms -Jeans Equation, Yukawa Potential, Escape Velocity, Dark Matter Halo

INTRODUCTION

Galaxies are huge collections of stars, planets, gas, and dust which are bound together by gravity. Each celestial body, whether a star, a stellar black hole, or a rogue planet, is bound to the centre of the galaxy in which they reside. In the galactic centre lies a supermassive black hole, usually millions of times more massive than our own star, the Sun. In other orbital systems, such as our solar system, the immense gravitational force exerted by our Sun causes everything in it to revolve around. However, in galaxies this remains untrue. In our Milky Way, the supermassive black hole at the centre, Sagittarius A*, contributes only 0.01 percent of the mass of the whole galaxy. Yet every celestial body revolving around the galactic centre does so with immense speeds. Why? This question led to a hypothesis of a new substance located within galaxies, known as dark matter. However, the particles which make up dark matter remain speculative to this very day.

In order to provide detailed hypotheses as to what subatomic particle dark matter is made up of, certain properties of said particle have to be calculated. These include the mass, the charge, and the spin. The former is the most important, as it influences dark matter's gravitational force. The lower bound of the mass (also referred to as the Gunn-Tremaine Bound, which applies only to fermionic particles with half-integer spins) can be easily derived after the Yukawa Force between the particles is calculated. Yukawa forces are the forces created by the force potential (denoted by V) within a set of particles, which requires the knowledge of three parameters. These are the force's magnitude (given by the standard formula denoted by Kepler), the force's range (denoted by the Greek letter Lambda, which is exponentiated against Euler's constant) and the force's strength relative to the inward gravity (denoted by the Greek letter Alpha).

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There are two ways for this to be done. One is the Pauli Exclusion Principle (which states that no two fermionic particles with half-integer spins can occupy the same quantum state). The other is Liouville's Theorem (which can only be used if the phase space distribution is known). Both methods are equally tangible, both concluding in different variants of the Jeans Equation. This paper will be using the former in its calculations, due to its simplicity and easy derivation of the escape velocity and fewer assumptions. This is done by modifying the standard escape velocity of a body via Newton's Shell Theorem, using the Jeans. This gives us an integral equation in terms of the radius of the dark matter halo to infinity. Different assumptions are used to calculate this, notably Alvey et al. 2.1.1, through the Zero Collision Boltzmann Equation combined with the Poisson Equation, This ultimately derives a formula in terms of the spherical tracer density (function $V(r)$) of the dark matter halo. This is made under the assumption that the Stellar component is in dynamic equilibrium inside the halo. textitDi Paolo et al. later assumes this to then perform further calculations, initally regarding the anisotropy as zero, then classifying it as a "nuisance parameter" (i.e. a parameter which alters the result of the equation, but doesn't have self-sustaining values). In their Jeans Analysis, the anisotropy is $1 - Vr/Vt$ (therefore the radial and tangential velocity dispersions are equal).

The identity of Dark Matter could solve many of astrophysics' most pondered
questions. It could explain in greater detail how the universe originated. This is due to the fact that over a quarter of CMBR (Cosmic Microwave Background Radiation) is emitted by matter with does not emit visible light. It could broaden our understanding of how the early universe was like, and how subatomic matter was first created. The high density of dark matter particles could explain how smaller fermionic particles bound together. The identity of dark matter could broaden our understanding of matter itself, about the different types of fermionic particles. It could help us understand the properties of more unknown particles, such as higher-mass fermions. It could help us understand how galaxies were formed, with each different arrangement of stars and other celestial bodies, how the stars revolve around the galactic centre with such speeds (this being one of the first questions to initiate the hypothesis of dark matter, binding stellar mass in galaxies). Solving Dark Matter would help us answer a multitude of questions about our universe and expand our understanding of both particle physics and astrophysics.

INTRODUCTION

This section will utilise the Jeans and aforementioned force potentials to obtain an integral equation for the lower bound of the mass of a dark matter particle. Here are the symbols for each quantity employed in this section:

Table 1: Table of quantities used in Section 2

Quantity Name	Quantity Symbol	Quantity Unit
Gravitational Potential	Φ_G	J/kg (2.1) and $J/M_\odot$ (2.2 to 2.4)
Density Function (2.1)	ρ	kg/m^{-3}
Density Profile (2.1)	v	kg/m^{-3}
Third-Dimensional Velocity	v^-	m/s
Stellar Anisotropy	β	*
Distance from Halo's Center to Chosen Body	r	pc
Halo Radius	R	pc
Halo Mass	M	$M_\odot$
Density Profile (2.2 to 2.4)	ρ	$M_\odot kpc^{-3}$
Yukawa Potential	Φ_Y	$J/M_\odot$
Yukawa Force Strength	α	*
Yukawa Force Range	λ	kpc
Net Potential	Φ_{net}	$J/M_\odot$
Escape Velocity	V_{esc}	m/s
Mass Profile	m	$M_\odot$
Lower Mass Bound	M_L	kg

The Jeans Equation

Before beginning this segment, it is important to clarify that the following calculations have been performed on the assumption of no net rotation (i.e. the radial and tangential velocities are equal) as was done in Di Paolo et. al's paper, and in Jo Bovy's textbook. To maintain continuity and ensure our process is easily understandable and replicable, we have followed their path and similarly regarded the net rotation to be zero.

The first step required to calculate the lower bound of the mass of a standard Dark Matter particle is through the Jeans Equation. The Jeans Equation is derived from the Collisionless Boltzmann Equation, also known as the Vlasov Equation. This equation is used to describe the behaviour of the distribution function of subatomic particles in motion under the influence of the gravitational force. The distribution is written in the form $f(x, v, t)$. Here, x and v represent Phase-space Cartesian Coordinates of the particles, and t represents the time in which the number of bodies in a differentiably small volume are present. This equation is combined with the Poisson Equation, which determines the gravitational field strength in a random mass distribution, in this scenario a phase-space distribution. This subsection will aim to review the role of the Jeans Equation by providing a brief derivation. The Poisson is then derived from the Laplace Equation, which is denoted by:

$$F(x) =- m\nabla\Phi_G(x)\#(1)$$

Where $\Phi_G(x)$ represents the Gravitational Potential. This equation is then rewritten by allocating the Laplace Operator to one side and substituting Newton's Law of Gravitation to obtain the following:

$$\nabla^2\Phi_G(x) = 4\pi G\rho(x)\#(2)$$

This equation can also be modified for a self-consistent system. The density function $\rho(x, t)$ can be integrated to obtain the Poisson in terms of $f(x, v, t)$, thus giving:

$$\nabla^2\Phi_G(x, t) = \int dvf(x, v, t)\#(3)$$

From here, it is important to note that the Collisionless Boltzmann Equation is applicable for any collisionless system, not just those of self-consistency. In terms of the previous Cartesian Coordinates, we would obtain:

$$\frac{\partial f(x,v,t)}{\partial t} + v\frac{\partial f(x,v,t)}{\partial x} + a\frac{\partial f(x,v,t)}{\partial v} = 0\#(4)$$

Since $a =- \frac{\partial\Phi_G}{\partial x}$, we could rewrite the equation as:

$$\frac{\partial f(x,v,t)}{\partial t} + v\frac{\partial f(x,v,t)}{\partial x} + \frac{\partial\Phi_G}{\partial x}\left(\frac{\partial f(x,v,t)}{\partial v}\right) = 0\#(5)$$

This results in the Collisionless Boltzmann equation, written in terms of x, v, t, and Φ_G. The Jeans can be obtained from this by integrating over v, which gives:

$$\int dv\frac{\partial f(x,v,t)}{\partial t} + \int dv(v)\frac{\partial f(x,v,t)}{\partial x} + \frac{\partial\Phi_G}{\partial x}\int dv\frac{\partial f(x,v,t)}{\partial v} = 0\#(6)$$

From here, we must take into account the density profile v and the third-dimensional velocity v^{-}, with these derivations respectively:

$$v(x) = \int dvf(x, v)\#(7)$$

$$v^{-}(x) = \frac{1}{v(x)}\int dv(v)f(x, v)\#(8)$$

From the previous integral equation, the final term disappears as f approaches zero (i.e., when v approaches ∞) thus resulting in:

$$\frac{\partial v(x)}{\partial t} + \nabla[v(x)\bar{v}(x)] = 0\#(9)$$

This is the continuity equation for the density $v(x)$. Thus, after integrating over v, we can multiply both equation (6) and equation (9) with the velocity component $\bar{v}_j$, and then subtracting the former from the latter, allows us to obtain the Jeans in Cartesian form.

$$v\left(\frac{\partial \bar{v}_j}{\partial t} + \bar{v}_i\frac{\partial \bar{v}_j}{\partial \bar{x}_i} + \frac{\partial \Phi_G}{\partial \bar{x}_j}\right) + \frac{\partial v\sigma_{ij}^2}{\partial \bar{x}_i} = 0\#(10)$$

This is the Jeans Equation in terms of rectangular Cartesian Coordinates. Substituting those with Polar Coordinates allows us to get:

$$\frac{d\left(v\overline{v_r^2}\right)}{dr} + v\left[\frac{d\Phi}{dr} + \frac{2\overline{v_r^2}-\overline{v_\theta^2}-\overline{v_\phi^2}}{r}\right]\#(11)$$

Substituting derived quantities, this equation can be simplified by substituting values such as the anisotropy β :

$$\beta = \frac{\overline{v_\theta^2}+\overline{v_\phi^2}}{2\overline{v_r^2}}\#(12)$$

And we can also substitute σ_r, in our equation, which is:

$$\sigma_r^2 = v\left(\overline{v_r^2}\right)\#(13)$$

Thus, resulting in the final product of:

$$\frac{d\sigma_r^2}{dr} + \frac{2\beta\sigma_r^2}{r} =- v\frac{d\Phi}{dr}\#(14)$$

This is the final Jeans Equation which will be used by this paper for further calculation. The Jeans Equation is used in various aspects of stellar kinematics. Here, we will be using to to calculate the density profile, which then can be used to calculate the force potentials using Newton's Shell Theorems.

The Gravitational Potential

To calculate the gravitational potential of the halo, we must use Newton's Shell Theorems.
The First Shell Theorem states that any body inside a uniform spherical shell experiences no net gravitational force from that shell.
The Second Shell Theorem states that any body outside a uniform spherical shell experiences a gravitational force such that all the shell's mass is concentrated at its centre.
Thus, the gravitational potential can be written as:

$$\Phi_G(r > R) =- \frac{GM}{r}\Phi_G(r < R) =- \frac{GM}{R}\#(15)$$

Where M is the mass of the shell, R is the radius of the shell, and r is the distance of the particle body from the shell's centre. In the context of dark matter halos, R represents the range of the phase-space distribution. We can take the mass of the shell to be it's volume $4\pi R^2$ multiplied by it's density $\rho(R)$, giving us:

$M = 4\pi G\rho(R)\Delta R\#(16)$

Thus, we can write $d\Phi_G$ as:

$d\Phi_G =- \frac{G}{r}4\pi G\rho(R)dR, r > R d\Phi_G =- \frac{G}{R}4\pi G\rho(R)dR, r < R\#(17)$

If we integrate each equation over all R, we will obtain the potential for all shells, which becomes:

$\Phi_G =- 4\pi G\frac{1}{r}\int_0^r R^2\rho(R)dR\Phi_G =- 4\pi G\int_r^\infty R\rho(R)dR\#(18)$

Thus, the final net gravitational potential becomes:

$\Phi_G =- 4\pi G\left[\frac{1}{r}\int_0^r R^2\rho(R)dR + 4\pi G\int_r^\infty R\rho(R)dR\right]\#(19)$

This will later be added to the Yukawa Potential for the halo to calculate the Escape Velocity for an internal body in Section 2.4.

The Yukawa Potential

The Yukawa Potential is the potential of the Yukawa Force, which is a force between two fermionic particles that mediates their exchange. The potential for a single particle is written as:

$\Phi_Y =- \alpha\frac{Gm_1m_2}{R}e^{-ar}\#(20)$

Here, α represents the overall strength of the force. For the sake of simplification, we have also defined the variable a as $1/\lambda$, where λ is the overall range of the force, expressed in parsecs (pc). For a spherical shell, the potential must be integrated. The dark matter halo itself can be treated as a concentric series of shells according to the Pauli Exclusion principle. Hence, according to this, the net Yukawa Potential can be expressed as:

$d\Phi_Y(R) = Asinh(aR)\frac{e^{-a(R-r)}}{r}, r > R\#(21)\ d\Phi_Y(R) = Asinh(ar)\frac{1}{r}, r \leq R\#(22)$

Here, A is a finite value obtained as r approaches zero (i.e. when the distance between the halo's centre and the body inside said halo approaches zero). $Asinh(aR)$ is the overall strength of the force, while the exponential e^{-ar} accounts for the fall-off outside the shell. If we assume that a approaches zero, this implies that e^{aR} approaches one $\left(\because e^0 = 1\right)$ and $sinh(aR)$ approaches zero ($\because sinh(0) = 0$). Hence,

$Asinh(aR) \rightarrow - GM\#(23)$

Where M is the mass of the halo. The mass can be substituted by the volume times the density:

$Asinh(aR) \rightarrow -\ G\left(4\pi R^2\rho(R)\Delta R\right)\#(24)$

Moving A to one side, we get:

$$A \rightarrow -\frac{4\pi\alpha GR^2\Delta R\rho(R)}{sinhaR}\#(25)$$

From here, we can substitute A in the Yukawa Potential to become the following for $R < r$, and $R > r$.

$$d\Phi_Y(R) = -\frac{4\alpha\pi G}{r}R^2\Delta R\rho(R)e^{aR-ar},\ r > R\#(26)\ d\Phi_Y(G) = -\frac{4\alpha\pi G}{sinh(aR)r}R^2\Delta R\rho(R)sinh(ar),\ r \leq R\#(27)$$

Thus, integrating over all values of R , we can obtain the final equation for the Yukawa Potential:

$$\Phi_Y(R) = \left[\frac{-4\alpha\pi G}{r}\right]\left[\int_0^r R\rho(R)e^{aR-ar}dR + \int_r^\infty \frac{R\rho(R)sinh(ar)dR}{sinhaR}\right]\#(28)$$

This is analogous to our expression in Section 2.2 for the gravitational force.

Escape Velocity and Gunn-Tremaine Bound

Adding the Gravitational Potential and the Yukawa Potential will allow us to get the Escape Velocity of a body within the dark matter halo. This is shown using:

$$\frac{V_{esc}^2}{2} = -\Phi_{net}(R) = -\left[\Phi_G(R) + \Phi_Y(R)\right]\#(29)$$

Where Φ_{net} represents the net potential, that is, the sum of the Gravitational and Yukawa Potentials. From our derivation in sections 2.2 and 2.3, we can write:

$$\frac{V_{esc}^2}{2} = 4\pi G[\left(\frac{1}{r}\int_0^r R^2\rho(R)dR + \int_r^\infty R\rho(R)dR\right) + \frac{\alpha}{r}\left(\int_0^r R^2\rho(R)e^{aR-ar}dR + \int_r^\infty \frac{R^2\rho(R)sinh(ar)dR}{sinh(aR)}\right)]\ (30)$$

Moving V_{esc}^2 to one side, we obtain:

$$V_{esc}^2 = 8\pi G[\left(\frac{1}{r}\int_0^r R^2\rho(R)dR + \int_r^\infty R\rho(R)dR\right) + \frac{\alpha}{r}\left(\int_0^r R^2\rho(R)e^{aR-ar}dR + \int_r^\infty \frac{R^2\rho(R)sinh(ar)dR}{sinh(aR)}\right)]\ (31)$$

This is the Escape Velocity of a body inside a dark matter halo. From here, we can use the Pauli Exclusion Principle (which states that no fermionic particles with half-integer spins may occupy the same space) to obtain an equation for the lower bound of dark matter, also called as the Gunn-Tremaine Bound. From the principle, it can be deduced that a self-gravitating body that completely consists of fermionic particles possesses a velocity derived from the mass and density profile. This is written as:

$$\vec{V} = \sqrt[3]{\frac{6\pi^2\rho(R)}{gm^4}}\#(32)$$

The bound can be obtained from the Pauli Exclusion Principle from the following condition:

$$M_L(R) \geq \sqrt[4]{\frac{6\pi^2 \rho(R)}{gV_{esc}(R)^3}} \#(33)$$

Substituting the escape velocity with that of the derived expression, we can obtain the final values for the lower bound.

RESULTS

Coding the Equation

After we have acquired the final equation for the lower bound, we can code this expression to obtain values for the escape velocity in terms of the range (λ) and the strength (α). Before we begin the calculations, we must import the radial and density profile values calculated by Alvey et. al in their code repository. The values are stored in the files output_rho.txt for each listed galaxy. In these files contain dictionaries for each galaxy. The first column represents the different radial values chosen by Alvey et. al, while the following columns contain the corresponding $\rho(r)$ values. The second column represents the midpoints of the density profile, and is labelled as "mid" in Alvey's code. The following columns are standard deviations "1su", "2su", and so on. Taking the values from each text file output.rho into variables:

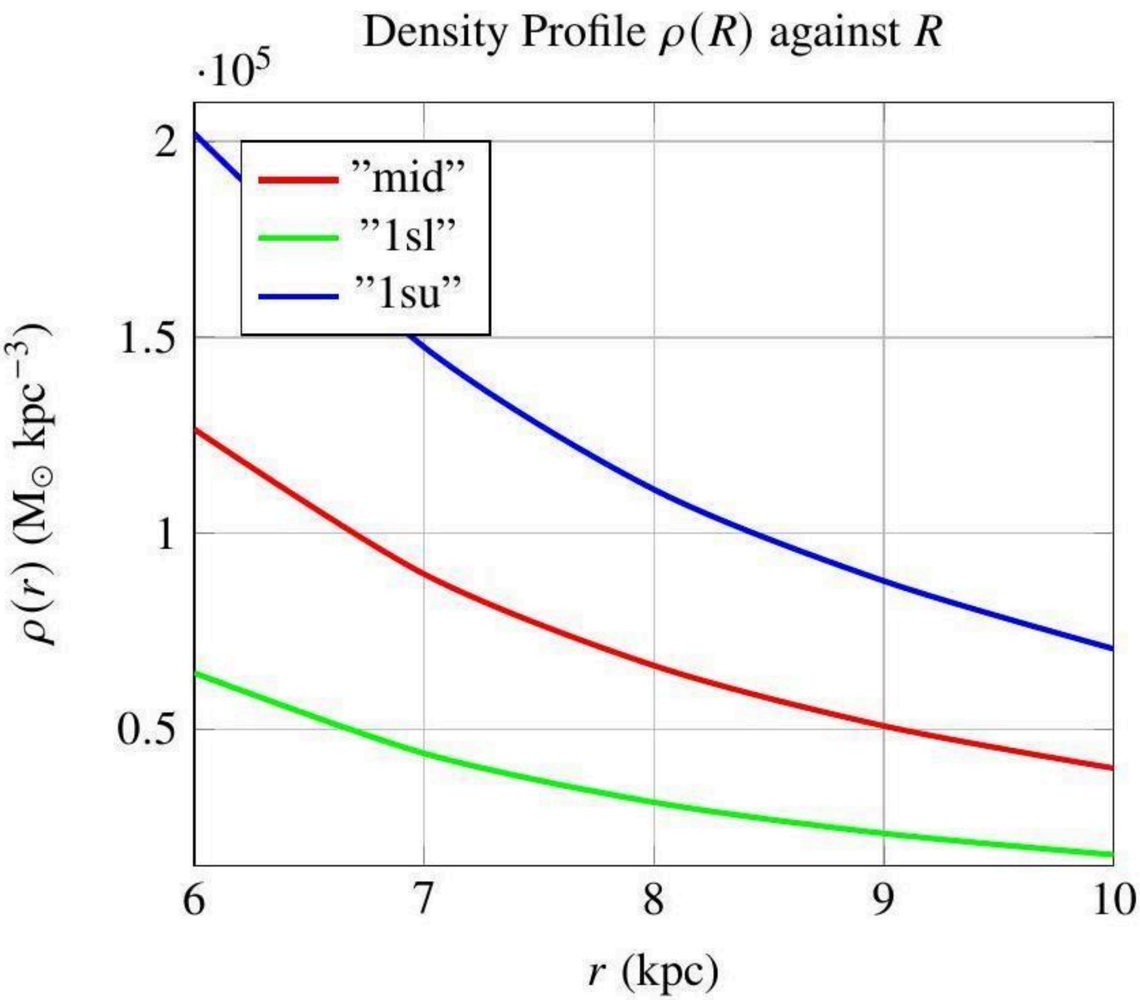

Figure 1: Graph illustrating the density profile of Leo I, with $\pm$ 1 sigma deviations

```
import numpy as np
r1 = np.loadtxt("/content/output_rho.txt", skiprows=1)
r2 = np.loadtxt("/content/output_rho.txt.1", skiprows=1)
r3 = np.loadtxt("/content/output_rho.txt.2", skiprows=1)
```

From these dictionaries, we must obtain the radial and density profile values individually from Alvey et. al's code. To do this, we create functions to obtain the values of r and rho (specifically the midpoints) from the text files. We can use the functions load_r(dwarf) and load_rho(dwarf) from Alvey et al.'s repository as references:

```
def load_r(r):


data = r
r_data = {'r': data[0, 1]}
return r_data
```

```
def load_rho(r):
data = r
rho_data = {'mid': data[0, 1]}
return rho_data
```

After obtaining the $\rho(r)$ values, we need to store the data in arrays for calculating the escape velocity:

```
galaxy_arr = ["LeoI", "LeoII", "Fornax"]
r_arr = [load_r(r1), load_r(r2), load_r(r3)]
rho_arr = [load_rho(r1), load_rho(r2), load_rho(r3)]
print(rho_arr)
alpha_arr = [10**16, 10**17, 10**18, 10**19, 10**20]
lambda_arr = [10**-2, 10**0, 10**2, 10**4, 10**6]
```

Here, we have taken r_arr as an array for the radial values of each galaxy. Likewise, we have taken rho_arr for the density profile values. The chosen values in alpha_arr and lambda_arr have been obtained from Bogorad et. al's Figure 1. According to Bogorad et. al,

$$\alpha \propto \lambda, \lambda > 10^3 kpc \# (34)$$

Thus, for higher order α, one must apply lower values for λ, and vice versa.

```
for vle in range(0, len(alpha_arr)):
    alpha = alpha_arr[vle]
    lambda1 = lambda_arr[vle]
    CTW = 4*3.14159*6.6743*10**-11
    CTY = alpha*lambda1/r
```

After this, we need to calculate the integrands of the gravitational and Yukawa forces. This is done using the Simpson's rule (python: scipy.integrate.simpson()) to calculate the integrands for all values in the arrays for the values of R and ρ (since each halo is defined as a series of shells, the different $\rho(r)$ values will differ depending on the value of R). The Heaviside's rule (python: np.heaviside() is used to mark the limits of the integrands (specifically the upper and lower limit of r to collect the values of $\rho(R)$. Each integrand for the Gravitational Potential can be defined as follows:

```
intg1 = (1/r) * (simpson(CTW * (dictr['r'])**2 * dictrho['mid'] *
    np.heaviside(r-dictr['r'], 1.0), x = dictr['r']))
intg2 = simpson(CTW * dictr['r'] * dictrho['mid'] * np.heaviside(dictr['r']-
    r, 1.0), x = dictr['r'])
```

However, the integrands of the Yukawa Potential require the specific radius of the halo to be taken to account. The reason for this is the values stored in output_rho.txt far exceed the natural dimensions of the typical halo (the values reaching to 500 kiloparsecs), although at those lengths the halo would begin to decay as the Gravitational
and Yukawa Forces are too weak to keep it bound. Hence, we will need to truncate the values in output_rho.txt:

```
for x in range(0,len(dictr['r'])):
    if dictr['r'][x] - r >= 0:
        break
r_arr_trnc_upper = dictr['r'][0:x]
r_arr_trnc_lower = dictr['r'][x-1:]
rho_arr_trnc_upper = dictrho['mid'][0:x]
rho_arr_trnc_lower = dictrho['mid'][x-1:]
```

Thus, the first Yukawa integrand can be written as:

```
inty1 = simpson(CTW * CTY * (r_arr_trnc_upper)**2 * rho_arr_trnc_upper *
    2.71418**(r_arr_trnc_upper/lambda1 - r/lambda1) * np.heaviside(r-
    r_arr_trnc_upper, 1.0), x = r_arr_trnc_upper)
```

However, there's still an issue. If one were to exponentiate x over a very large number over a certain value, python would register said value as infinity, and thus the net result would approach infinity as well. If we were to exponentiate over the inverse of a very large number, it would approach zero. The second Yukawa integrand uses the hyperbolic sine, which is written as:

$$sinh(x) = \frac{e^{x} - e^{-x}}{2} \#(35)$$

Thus, if x were a very large number, $sinh(x)$ would be infinity and the integrand invalid. Hence, the equation needs to be rewritten in negative exponents:

$$\frac{sinh(x)}{sinh(y)} = \left(e^{x-y}\right)\frac{1-e^{-2x}}{1-e^{-2y}} \#(36)$$

Writing this for the integrand in python gives us:

```
inty2 = simpson(CTW * CTY * (r_arr_trnc_lower)**2 * rho_arr_trnc_lower *
    ((2.71418**(y-z))*(1-2.71418**(-2*y))/(1-2.71418**(-2*z))) *
    np.heaviside(r_arr_trnc_lower-r, 1.0), x = r_arr_trnc_lower)
```

After deriving the integrands, and choosing a suitable value for r (i.e. the distance between the body and the halo's centre), we get the final code for the escape velocity V_{esc} of the body. The full repository can be found here

Running the Code

To test our code, we can use a sample of the radius and density profile value for three galaxies (Leo I, Leo II, and Fornax), a chosen value for the radius of integration limit R (say 10), alongside five chosen values of α and λ, chosen with respect to Bogorad et al.'s Yukawa derivation, as mentioned previously. One must have 3 ready values for calculation (as mentioned in the coding section); α, λ, and r. The radius and density profile values (R and $\rho(R)$) have been given in the code repository in Alvey et al.'s paper in the output_rho.txt files for each galaxy. After we input the five values of α while keeping λ and r constant (let's say $\lambda = 10^{-5}$ parsecs to match with the bounds in Bogorad et. al, and $r = 0.1$ parsecs to match with the radial values in Alvey et. al's repository, for example) we get:

Table 2: Escape Velocity for LeoI, α modified $\left(\lambda = 10^{-6}\right)$

$\alpha(*)$	$V_{esc}\left(m\, s^{-1}\right)$
10^{16}	45405.508
10^{17}	50294.028
10^{18}	84898.642
10^{19}	232358.950
10^{20}	722370.889

Table 3: Escape Velocity for LeoII, α modified $\left(\lambda = 10^{-6}\right)$

$\alpha(*)$	$V_{esc}\left(m\, s^{-1}\right)$
10^{16}	31050.746
10^{17}	40145.409
10^{18}	89926.104
10^{19}	269883.691
10^{20}	848729.775

Table 4: Escape Velocity for Fornax, α modified $\left(\lambda = 10^{-6}\right)$

$\alpha\left(^{*}\right)$	$V_{esc}\left(m\, s^{-1}\right)$
10^{16}	83771.324
10^{17}	84341.559
10^{18}	89845.076
10^{19}	132885.392
10^{20}	336930.891

We can also perform vice versa, where α remains constant and λ is modified:

Table 5: Escape Velocity for LeoI, λ modified $\left(\alpha = 10^{4}\right)$

$\lambda (kpc)$	$V_{esc}\left(m\, s^{-1}\right)$
10^{-2}	522273.689
10^{0}	3201899.835
10^{2}	4380168.135
10^{4}	4483038.760
10^{6}	4483060.382

Table 6: Escape Velocity for LeoII, λ modified $\left(\alpha = 10^{4}\right)$

$\lambda (kpc)$	$V_{esc}\left(m\, s^{-1}\right)$
10^{-2}	606648.558
10^{0}	2612206.158
10^{2}	2972895.180
10^{4}	2986898.989
10^{6}	2986902.214

Table 7: Escape Velocity for Fornax, λ modified $\left(\alpha = 10^{4}\right)$

$\lambda (kpc)$	$V_{esc}\left(m\, s^{-1}\right)$
10^{-2}	251243.158
10^{0}	3589760.813
10^{2}	7733458.096
10^{4}	8371041.096
10^{6}	8371179.145

DISCUSSION

The modifications of α and λ

Based on the final results on the values of the escape velocity and the lower bound, we can compare the relationship between α and λ and the escape velocity.

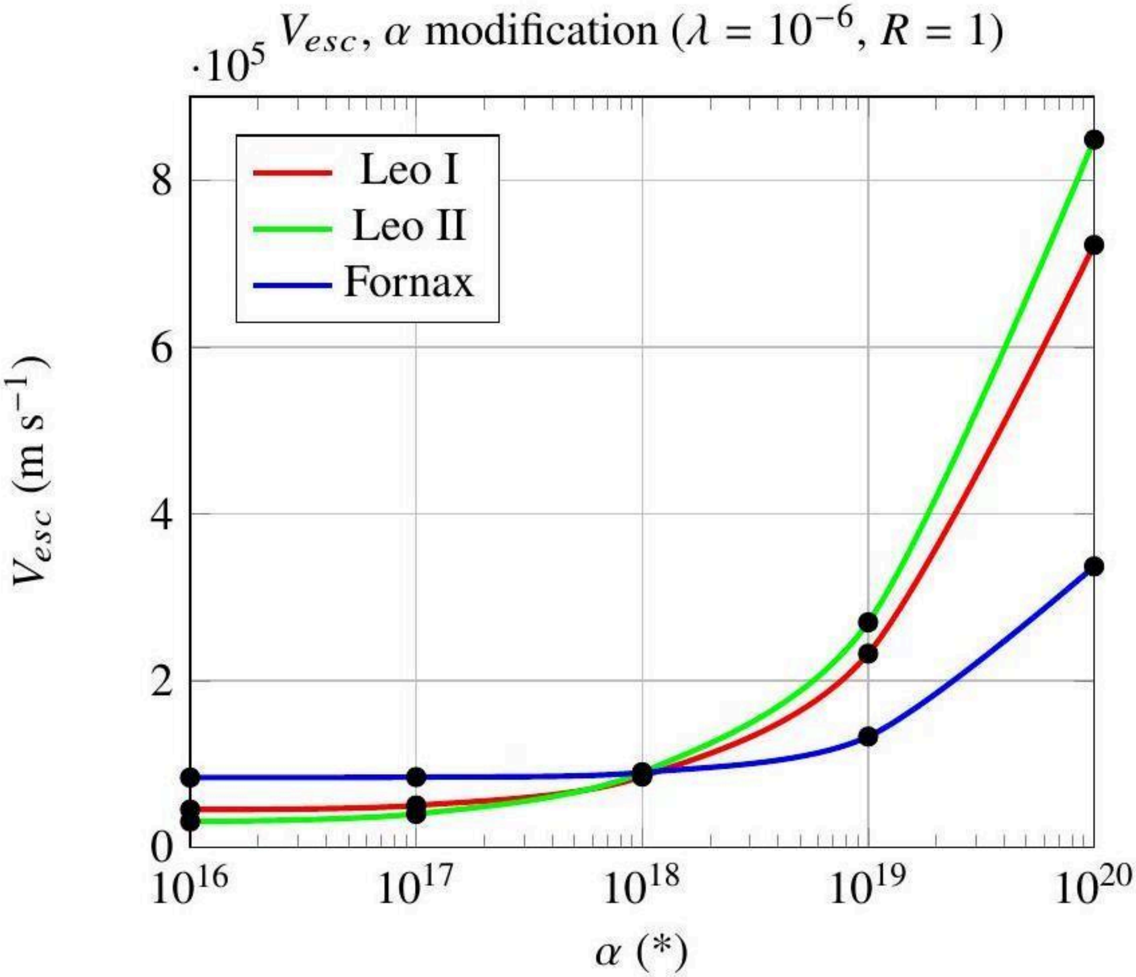

Figure 2: Graph illustrating the change in V_{esc} of a dark matter halo in Leo I, Leo II, and Fornax by modifying $\alpha\left(\lambda = 10^{-6}\right)$

Based on the gradients of the graph, V_{esc} follows a quasi-parabolic curve. However, as the value of r and λ change, the net difference between the escape velocities also change. Keeping r as 0.1 will cause the escape velocity to become too small for python to register, hence we ought to take a larger value, let's say $R = 1$. Here, The values of α would cause negligible change to the escape velocity when $\lambda < 10^{-4}$:

Table 8: Escape Velocity for LeoI, α modified $\left(\lambda = 10^{-5}\right)$

$\alpha(*)$	$V_{esc}\left(m\, s^{-1}\right)$
10^{-2}	39360.29989023125
10^{0}	39360.29989023134
10^{2}	39360.299890232236
10^{4}	39360.29989024127
10^{6}	39360.29989033159

Before 10^{-5}, the change in the escape velocity remains almost constant with respect to the values, with the succeeding value being approximately 3.16 x more than the preceding. From 10^{-5} onwards, the change in the escape velocity approaches zero at a rapid rate. While substituting λ with decimal exponents of 10 between -4 and -5 showcases the rate at which the change in V_esc decreases. Hence, comparing α with V_{esc} and λ with the change of V_{esc} :

$V_{esc} \propto \alpha^2 \#(37)\ \Delta V_{esc} \approx k,\ \lambda \in 10^{Z}: \lambda > 10^{-4} \#(38)$

Another factor to consider is the value of r. When $r \leq 0.01, r \leq r_{min}$, where r_{min} is the minimum value of r in Alvey's dictionary. Thus, there are no real values of V_{esc} at this range of values of r. However, as r increases, the change in V_{esc} likewise approaches zero at a rapid rate. When $\lambda = 10\ **-\ 4$, the change in V_{esc} reaches complete nil:

$\Delta V_{esc} \approx k,\ R \in 10^{Z}: R > 10^{1} \#(39)$

However the limiting values of both r and λ are interdependent. Decreasing λ will cause ΔV_{esc} to reach zero at a larger r value, and vice versa.
If we execute the same set of code while switching the roles of α and λ (i.e. α remains constant and λ varies), this is what we would obtain if $\alpha = 10^{-5}$:

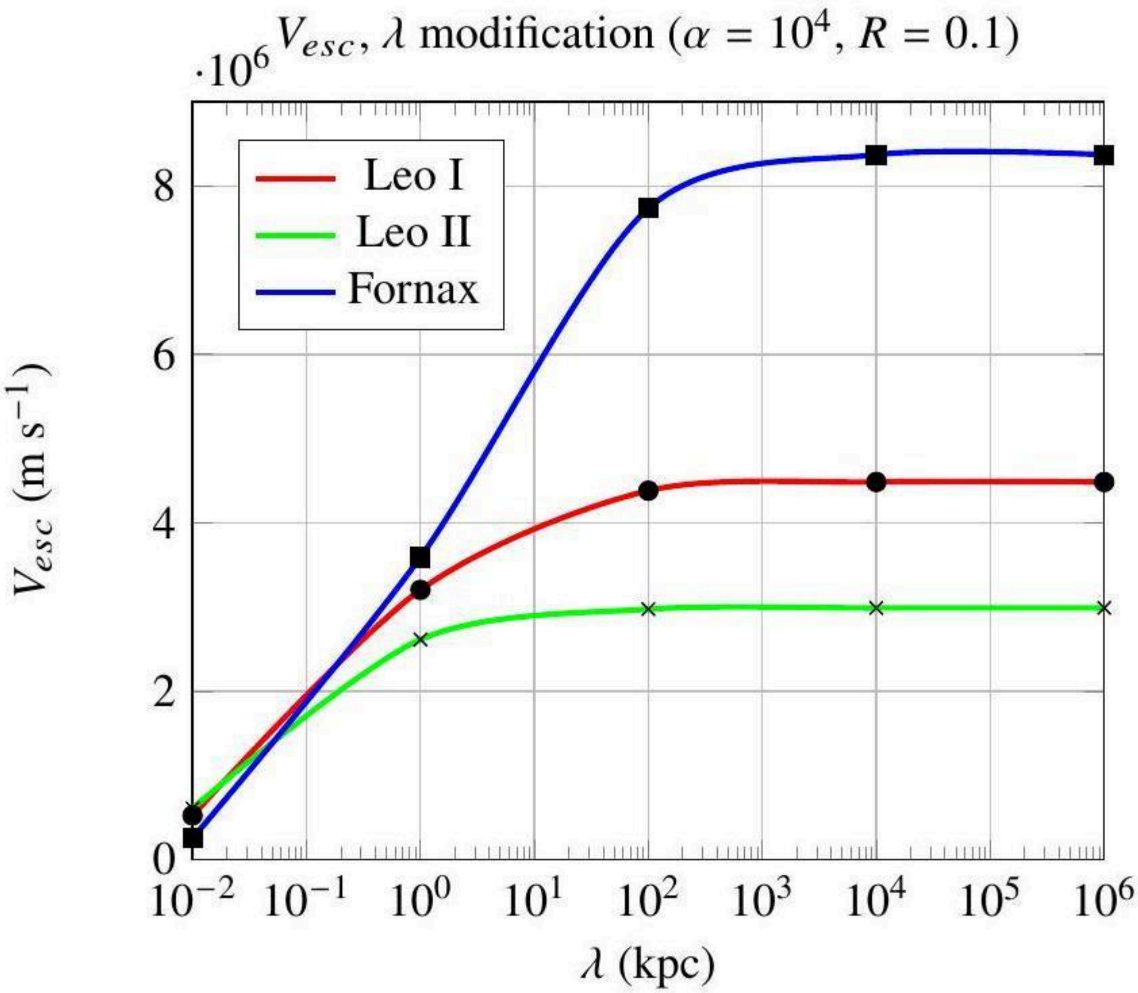

Figure 3: Graph illustrating the change in V_{esc} of a dark matter halo in Leo I, Leo II, and Fornax by modifying $\lambda(\alpha = 10^4)$

Based on the gradients of the graph, V_{esc} follows an asymptotic curve. An asymptotic curve is a curve that slowly approaches a line but takes an infinite amount of length to reach it. Here, each galaxy has it's own range limit, for which after that limit has been crossed, V_{esc} remains constant. Before this limit, V_{esc} initially increases at a constant rate. Thus, it can be concluded that:

$$V_{esc} \propto \lambda,\ \lambda > \lambda_{lim}\ V_{esc} \approx k,\ \lambda > 10^4 \#(41)$$

Just like α, λ is also interdependent with R .

The Gunn-Tremaine Bound

To calculate the Gunn-Tremaine Bound, we will have to add some extra bits of code into our repository. Initially, we will have to define a function that returns the mass bound:

```
def get_gtb(galaxy,dictr,dictrho,r,alpha,lambda1):
```

Now, since the formula for the bound requires the density profile $\rho(r)$, we will need to extract it from Alvey et. al's repository, based on the distance between the body and the halo's centre r :

```
rho_SI = None
    for x in range(0,len(dictrho['mid'])):
        if dictr['r'][x] - r >= 0:
            if dictr['r'][x] - r > r - dictr['r'][x-1]:
                    rho_kpc = dictrho['mid'][x-1]
                rho_SI = rho_kpc * 6.7696 * 10**-29
            else:
                rho_kpc = dictrho['mid'][x]
                rho_SI = rho_kpc * 6.7696 * 10**-29
            break
```

Afterwards, we can safely code the formula to calculate the escape velocity:

```
vesc_val = get_vesc(galaxy,dictr,dictrho,r,alpha,lambda1)
Mass_Bound = ((6 * 3.14159**2 * rho_SI)/(2 * vesc_val**3))**0.25
print(Mass_Bound)
```

Henceforth, we have obtained the final code for calculating the Lower Mass Bound, with custom α and λ values. The same repository here can be used to thoroughly calculate the lower mass bound M_L in kilograms.

To convert these values from SI units to electronvolts, we need to use the famous Theory of Relativity:

$$E = mc^2 \#(42)$$

From this, we can calculate the Joule energy of 1 kg to be 9×10^{16}. Then, we can divide this by the elementary charge e to obtain 5.6×10^{35} electronvolts per every kilogram.

CONCLUSION

To conclude, we have calculated the escape velocity of a body inside a dark matter halo (V_{esc}) through two forces present between the particles: the Gravitational force, and the Yukawa force, the latter dependent on the net strength of the force (α) and the range of the force (λ). After coding the final integral equation, we can obtain values of V_{esc}, as well as the Gunn-Tremaine Bound M_L. These values can be modified by changing α, λ, and R and can thus derive a pattern as to how the mass of dark matter halos are reliant on these two parameters through V_{esc}. Based on our findings, we have concluded that α and λ follow quasi-parabolic and asymptotic relationships with V_{esc} respectively, where λ is directly proprotional to the escape velocity until it hits a certain limit λ_{lim}.

The Yukawa constraints (i.e. α and λ) for the net potential are listed as valid magnitudes given in Bogorad et al.'s Figure 1 describing self-interactions between the two, allowing for accurate measurements of the Yukawa potential. This potential for the halo can be converted into an integrand analogous to Jo Bovy's Equation 2.23 regarding the net gravitational potential for a series of shells. This permits for V_{esc} to be calculated through the Law of Conservation of Energy. Thus, we have been able to find the relationship between the Escape Velocity V_{esc} of a body inside a dark matter halo, and the Yukawa constrains for the force inside the halo. After taking sample values from Alvey et. al's repository, particularly the values in output_rho.txt, to finally obtain suitable escape velocities for Leo I, Leo II, and Fornax.

After calculating the escape velocity, we have been able to find the Lower Mass Bound M_L (also called the Gunn-Tremaine Bound) using Alvey et. al's Equation 9 to calculate this bound, and write it in terms of electronvolts.

REFERENCES

[1] Alvey et al., "New Constraints on the Mass of Fermionic Dark Matter from Dwarf Spheroidal Galaxies", 2020, p. 1-4
[2] Bogorad et al., "Coherent Self-Interactions of Dark Matter in the Bullet Cluster", 2025, p. 3, 8
[3] Di Paolo et al., "Phase Space Mass Bound for Fermionic Dark Matter from Dwarf Spheroidal Galaxies", 2018, p. 4-5
[4] Jo Bovy, "Dynamics and Astrophysics of Galaxies", chap. 2.1, 2.2, 5.3, 5.4

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