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A Formal Decomposition of the Partition Function at $\frac{n}{2}$

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ABSTRACT

This study presents the derivation of a decomposition of the Partition Function $p(n)$ for $n \in W$. It constitutes partitioning all partitions into two distinct classes, which is obtained by decomposing it at $\lfloor \frac{n}{2} \rfloor$

In order to find the number of partitions in the set containing the largest part $< \lfloor \frac{n}{2} \rfloor$, we implement the usage of restricted partitions $p_{a,b}(m)$ representing the number of partitions of m with each part λ_i bound by inequalities of being $a \le \lambda_i \le b$. These methods result in the formation of separate unique definite summations, which upon addition yields the value of the Partition Function $p(n)$.

This combinatorial concept for understanding partition functions proposes a simple yet unique perspective to the working of Partition Functions. The formalization of this offers a structured image of the identity presented in the paper, making it convenient and comprehensible to the readers.

INTRODUCTION

Number Theory is one of the fundamental branches of mathematics which offers insight into properties, structures, arithmetic functions and behaviors of numbers. Partition Functions is one such concept which resides in this branch of mathematics. This function can be defined as the number of partitions in which a whole number can be expressed as the sum of positive integers independent of the order of summation.

Partition Function has been a field of study in mathematics that has gained significant attraction and contribution from well known personalities such as Leonhard Euler, Ramanujan and G. H. Hardy. Its significance in being a crucial foundation in number theory is due to the diversity of perspectives it offers on concepts such as generating functions, asymptotics, patterns, congruences, combinatorial sequences and advanced theory [4].

A crucial component of this paper is built upon the concept of Restricted Partitions [4] $p_{a,b}(m)$ denotes the number of partitions of m such that each part lies between a and b; that is, $\forall p(m) \in N$ and $m \in W$.

$$m = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_k$$

We require,

$$a \le \lambda_i \le b \; \forall i \in [1, k]$$

This paper presents a combinatorial interpretation of Partition Functions based on the decomposition of partitions into two parts. Existing mathematical literature generally expresses Partition Functions through asymptotes, infinite summations, infinite products and modular functions derived through distinct routes.

In contrast, this work presents a decomposition identity expressed as the result of a non-asymptotic finite summation. This not only provides convenience in understanding the concept of this function at its core but also guides readers in understanding Partition Functions through a counting-based approach.

METHODS

Setup and Preliminaries

We begin by denoting $p(n)$ as the number of partitions of any whole number n. Let $p_{a,b}(m)$ denote the number of partitions of a whole number m such that each part lies between a and b where $a, b \in W$. Any Partition of n may be denoted as

$$n = \sum_{i=1}^{k} \lambda_i \; \forall n \in W$$

where λ_i represents the i-th part of the partition and k represents the number of parts in the partition. Furthermore, we assume

$$\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \dots \lambda_k$$

DECOMPOSITION AND CONSTRUCTION

Decomposition via Midpoint Split

The midpoint split method presented in this paper introduces the distinction between two sets of partitions for n. The distinction between the two sets is made based on the largest part of the partition being $\ge \lfloor \frac{n}{2} \rfloor$ and $< \lfloor \frac{n}{2} \rfloor$, where the floor function ensures consistency for odd and even values of n. The reason for this distinction lies in the fundamental constraint that parts of any partition must sum to n.

Analysis of Partitions with Largest Part $\ge \lfloor \frac{n}{2} \rfloor$

For any partition with the largest part being $\geq \lfloor \frac{n}{2} \rfloor$, the remainder r after removing the largest part would be bound by the inequality of being $\leq \lfloor \frac{n}{2} \rfloor$. Therefore, the largest part of any partition can appear at most twice when it is equal to $\lfloor \frac{n}{2} \rfloor$ and only once when it is greater than $\lfloor \frac{n}{2} \rfloor$. We can present this inference as

$$\lambda_1 > \frac{n}{2} \tag{1}$$

$$\Rightarrow 2\lambda_1 > n \tag{2}$$

Since the sum of all parts of a partition of n cannot be greater than the integer n itself, we can conclude that for all $\lambda_1 > \frac{n}{2}$, there cannot be more than one occurrence of λ_1. For any partition of n with largest part λ_1, there exists a remainder integer r which can further be expressed as individual partitions of its own. We can express this as $p(r)$. Additionally, we can also infer that each partition of r uniquely extends to a partition of n by adjoining the part λ_1, leading to an increase
in the number of partitions of n.

Throughout all the partitions which r extends to for largest part λ_1, the only part that remains constant is λ_1. Therefore, we can determine the number of partitions with largest part $\lambda_1 > \frac{n}{2}$ by yielding the number of partitions of remainder r where $r = n - \lambda_1$.

Due to this behavioral pattern, we can infer that the number of partitions with the largest part $\geq \lfloor \frac{n}{2} \rfloor$ is equal to the sum of the number of partitions of the remainders r such that $r = n - \lambda_1$ and $r \leq \lfloor \frac{n}{2} \rfloor$. Let us define this inference as

$$x = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p(i) \tag{3}$$

Analysis of Partitions with Largest Part < $\lfloor \frac{n}{2} \rfloor$

Unlike Partitions with the largest part $\geq \lfloor \frac{n}{2} \rfloor$, the behavioral pattern of those with the largest part $< \lfloor \frac{n}{2} \rfloor$ is quite contrasting. For any such partition that contains the largest part as λ_1,

$$\lambda_1 < \lfloor \frac{n}{2} \rfloor$$

$$\Rightarrow 2\lambda_1 < n$$

Therefore, parts like λ_1 may appear two or more times in such partitions, unlike the case in the previously discussed subsection where the largest part = $\lfloor \frac{n}{2} \rfloor$ which yields a partition as the sum of two equivalent terms. Due to this repetitive occurrence of terms, we will need to yield this second set of partitions through an alternative approach.

In order to evaluate the second set, we use restricted partitions. We define a function which obtains the desired number of partitions under specific constraints on the partitions. We have previously defined this function as $p_{a,b}(m)$ with parameters $a, b, m \in W$.

The use of restricted partitions helps determine the number of partitions of the remaining integer after removing the largest part. We define this as $p(r)$ for each partition where $r = n - \lambda_1$. However, due to the constraint $\lambda_1 \geq \lambda_2$ for all partitions of n, not all partitions of $p(r)$ extend to valid partitions of n. Therefore we must impose few constraints upon the partitions of r. To resolve this, we define each restricted partition as

$$p_{1,\lambda_1}(r)$$

Similar to the methodology applied to the number of partitions of n where $\lambda_1 \geq \lfloor \frac{n}{2} \rfloor$, there may be multiple partitions of $p_{1,\lambda_1}(r)$ which extend to produce valid partitions of n. Additionally, the only part that remains constant throughout the distinct partitions of n extended only by partitions of a remainder integer r is λ_1. Hence, the number of partitions with the largest part λ_1 can be considered equivalent to the number of restricted partitions $p_{1,\lambda_1}(r)$ where $r = n - \lambda_1$ is the remainder integer of λ_1.

Therefore, to determine the total number of partitions that require this constraint, carry out a definite summation of this restricted partition represented by y. Moreover, since the value of r in the partitions of n may change per partition, we define a variable i in place of r.

This way, we can yield the number of partitions of r which satisfy given constraints for every partition's largest part λ_1. Therefore, we can conclude that the value of y can be determined by the following formulation:

$$y = \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} p_{1,n-i}(i) \tag{4}$$

Reconstruction of $p(n)$ via Largest Part Split

In the two latest subsections, we determine the values of both sets obtained from the initial decomposition of $p(n)$. Therefore, we have explicit expressions for both x and y. Thus,

$$p(n) = x + y \tag{5}$$

Substituting (3) and (4) into (5), we obtain

$$p(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p(i) + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} p_{1,n-i}(i) \tag{6}$$

This completes the reconstruction of $p(n)$ via the largest part split. This Decomposition Identity expresses $p(n)$ as the sum of two finite summations from partitions with the largest part above and below the midpoint.

DISCUSSIONS

In order to verify our derivation, we will determine $p(n)$ with the decomposition identity for different values of n. Let us verify this identity with the help of two distinct cases with $n = 5$ and $n = 6$ in order to validate it for both odd and even integers. The results obtained are compared with known results of $p(n)$ to ensure correctness and consistency. We also consider both odd and even integers to ensure the applicability of the identity. The following examples demonstrate the efficiency of the decomposition identity derived for determining partition values.

Verification of the Identity

Case 1 for Odd Numbers:

For $n = 1$:

Partitions of *n* = 1
1

Substituting the value of n, we get the bounds of summation as

$$\lfloor \frac{n}{2} \rfloor = \lfloor 0.5 \rfloor = 0 \tag{7}$$

Substituting (7) into the decomposition identity, we can express $p(n)$ for $n = 1$ as

$$p(1) = \sum_{i=0}^{0} p(i) + \sum_{i=1}^{1} p_{1,1-i}(i) \tag{8}$$

$$= p(0) + p_{1,0}(1) \tag{9}$$

$$= 1 + 0 \tag{10}$$

$$\therefore p(1) = 1 \tag{11}$$

For $n = 5$:

Partitions of $n = 5$
5
4 + 1
3 + 2
3 + 1 + 1
2 + 2 + 1
2 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1

Substituting the value of n, we get the bounds of summation as

$$\lfloor \frac{n}{2} \rfloor = \lfloor 2.5 \rfloor = 2 \tag{12}$$

Substituting (12) into the decomposition identity, we can express $p(n)$ for $n = 5$ as

$$p(5) = \sum_{i=0}^{2} p(i) + \sum_{i=3}^{5} p_{1,5-i}(i) \tag{13}$$

$$= p(0) + p(1) + p(2) + p1,2\,(3) + p1,1\,(4) + p1,0\,(5) \tag{14}$$

$$= 1 + 1 + 2 + 2 + 1 + 0 \tag{15}$$

$$\therefore p(5) = 7 \tag{16}$$

Case 2 for Even Numbers:

For $n = 4$:

Partitions of n = 4
4
3 + 1
2 + 2
2 + 1 + 1
1 + 1 + 1 + 1

Substituting the value of n, we get the bounds of summation as

$$\lfloor \frac{n}{2} \rfloor = \lfloor 2 \rfloor = 2 \tag{17}$$

Substituting (17) into the decomposition identity, we can express $p(n)$ for $n = 4$ as

$$p(4) = \sum_{i=0}^{2} p(i) + \sum_{i=3}^{4} p_{1,4-i}(i) \tag{18}$$

$$= p(0) + p(1) + p(2) + p_{1,1}(3) + p_{1,0}(4) \tag{19}$$

$$= 1 + 1 + 2 + 1 + 0 \tag{20}$$

$$\therefore p(4) = 5 \tag{21}$$

For $n = 6$:

Partitions of n = 6
6
5 + 1
4 + 2
4 + 1 + 1
3 + 3
3 + 2 + 1
3 + 1 + 1 + 1

2 + 2 + 2
2 + 2 + 1 + 1
2 + 1 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1 + 1

Substituting the value of n, we get the bounds of summation as

$$\lfloor \frac{n}{2} \rfloor = \lfloor 3 \rfloor = 3 \tag{22}$$

Substituting (22) into the decomposition identity, we can express $p(n)$ for $n = 6$ as

$$p(6) = \sum_{i=0}^{3} p(i) + \sum_{i=4}^{6} p_{1,6-i}(i) \tag{23}$$

$$= p(0) + p(1) + p(2) + p(+ p_{1,2}(4) + p_{1,1}(5) + p_{1,0}(6) \tag{24}$$

$$= 1 + 1 + 2 + 3 + 3 + 1 + 0 \tag{25}$$

$$\therefore p(6) = 11 \tag{26}$$

Case 3 for Large Numbers:

Let us verify the decomposition identity for largest values of n such as that of 8 in order to check its consistency.

For $n = 8$:

Partitions of n = 8
8
7 + 1
6 + 2
6 + 1 + 1
5 + 3
5 + 2 + 1
5 + 1 + 1 + 1

$4 + 4$
$4 + 3 + 1$
$4 + 2 + 2$
$4 + 2 + 1 + 1$
$4 + 1 + 1 + 1 + 1$
$3 + 3 + 2$
$3 + 3 + 1 + 1$
$3 + 2 + 2 + 1$
$3 + 2 + 1 + 1 + 1$
$3 + 1 + 1 + 1 + 1 + 1$
$2 + 2 + 2 + 2$
$2 + 2 + 2 + 1 + 1$
$2 + 2 + 1 + 1 + 1 + 1$
$2 + 1 + 1 + 1 + 1 + 1 + 1$
$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

Substituting the value of n, we get the bounds of summation as

$$\lfloor \frac{n}{2} \rfloor = \lfloor 4 \rfloor = 4 \tag{27}$$

Substituting (27) into the decomposition identity, we can express $p(n)$ for $n = 8$ as

$$p(8) = \sum_{i=0}^{4} p(i) + \sum_{i=5}^{8} p_{1,8-i}(i) \tag{28}$$

$$= p(0) + p(1) + p(2) + p(3) + p(4) + p_{1,3}(5) + p_{1,2}(6) + p_{1,1}(7) + p_{1,0}(8) \tag{29}$$

$$= 1 + 1 + 2 + 3 + 5 + 5 + 4 + 1 + 0 \tag{30}$$

$$\therefore p(8) = 22 \tag{31}$$

Therefore, we can conclude that the identity correctly determines the values of $p(n)$ for $n = 1, n = 4, n = 5, n = 6$ and $n = 8$ as $1, 5, 7, 11$ and 22 respectively. This agrees with known results of $p(n)$, hence verified.

Historical Relevance and Comparison

Mathematical literature has provided us with expressions of $p(n)$ in multiple forms including generating functions, identities, formulas, etc. One of the biggest contributions made in this field of mathematics was by Leonhard Euler. His work on partition functions has provided significant insight into the behavioral patterns of partition functions. One of the most commonly stated expressions is the Euler's generating function [1]:

$$\sum_{n=0}^{\infty} p(n)x^{n} = \prod_{k=1}^{\infty} \frac{1}{1-x^{k}}$$

which provides a non-asymptotic value of $p(n)$ upon carrying out an infinite summation and product. Similarly, the Hardy-Ramanujan formula [2]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{2\pi\sqrt{\frac{n}{6}}}$$

provides us with an asymptotic indefinite value of $p(n)$ which is highly efficient for finding estimated values of $p(n)$ for very high values of n. Moreover, unlike Euler's generating function and the Hardy-Ramanujan formula, Rademacher's formula [3]:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_{k}(n)\sqrt{k}\frac{d}{dn}\left[\frac{1}{\sqrt{n-\frac{1}{24}}} sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)\right]$$

is an exact convergent series representation for the integer partition function $p(n)$ with immense complexities due to the involvement of concepts such as roots of unity and the structure of the formula.

Mathematical literature presented by Indian mathematician Srinivasa Ramanujan on partition functions has provided us with insight into its behavioral patterns using the congruence formulas:

$$p(5n + 4) \equiv 0 \ (mod 5)$$
$$p(7n + 5) \equiv 0 \ (mod 7)$$
$$p(11n + 6) \equiv 0 \ (mod 11)$$

These formulae show divisibility patterns of partition functions. Additionally, their computation and structure primarily describe arithmetic patterns unlike recursive and generating-function-based formulations that we have previously viewed.

In contrast, the identity derived in this paper provides a simple structured nonasymptotic finite summation of known partitions which are separated at midpoint using combinatorics. Based on the major formulations and identities discussed thus far, the structure, computation and combinatorial reasoning do not resemble that of the decomposition identity derived in this paper. To the best of the author's knowledge, an explicit midpoint-split-based decomposition identity similar to the one derived in this paper has not been established in existing mathematical literature.

Limitations and Scope for Future Developments Despite the precision and accuracy of this decomposition identity, the limitations of the identity arise when the requirement of having prior knowledge of partition functions for integers lesser than n is needed.

Although the identity derived in this paper is recursive in nature due to its dependency on previously known values of $p(n)$, its primary importance lies in its structural framework and combinatorial interpretation rather than computational efficiency.

Unlike existing recursive formulations such as Euler's recursive formulation of partition functions, the identity derived emphasizes on the decomposition of partitions at midpoint and the influence of restricted partitions on partition functions.

However, future work may explore whether similar decomposition identities can be obtained through alternative thresholds beyond $\lfloor \frac{n}{2} \rfloor$. Additionally, due to the involvement of restricted partitions in the decomposition identity, any future developments made in order to derive values and expressions of restricted partitions can improve the structural and combinatorial understanding of partition functions.

CONCLUSION

This paper presents a formal decomposition identity which yields a finite nonasymptotic value for partition functions using combinatorics. This identity has been verified for both even and odd numbers. Its distinction lies in the application of the midpoint split which separates partitions based on the largest parts.

Furthermore, the structure of the identity presents the algebraic sum of two finite summations of partition functions as well as restricted partitions. Additionally, it presents insight into the role of restricted partitions on determining the value for any partition function.

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