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The Combinatorial Structure of Special Relativity

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ABSTRACT

In this paper, we investigate the rise of elementary symmetric polynomials in compositions under the hyperbolic tangent function. We first split the composition into two cases, where we examine the structure when an odd number and an even number of elements are composed. Through the use of floor functions, we unify the two cases, resulting in a closed form generalization. We prove this structure using mathematical induction.

Our contribution lies in applying this structure to special relativity, where we convert rapidities into velocities. The structure is valid for only collinear velocity compositions in 1 dimension. We then further prove that the structure obeys the postulates of special relativity by showing that it does not exceed the speed of light.

In addition, we develop a generalized formula for the total effect of the Lorentz factor for any number of collinear or parallel boosts. A boost is a transformation of coordinates between two inertial coordinate systems moving at constant relative velocity to one another without rotation.

In a purely theoretical scenario, we implement the structure to analyze the sufficient conditions by which the Lorentz factor converges for an infinite number of velocity additions. Finally, we derive generalizations of the Lorentz transformations.

INTRODUCTION

In classical mechanics, velocity addition is linear. Specifically, the resultant velocity, ω_i after i additions would be as the following:

$$\omega_i = \sum_{j=1}^{i} v_j .$$

However, this assumption breaks down when velocities approach the speed of light, since they exceed it. As a result, in his famous paper, "On the Electrodynamics of Moving Bodies," Einstein introduces a

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non-linear velocity addition formula.[1] This formula is consistent with the postulates of special relativity by ensuring that no composition of velocities exceed the constant speed of light, c.

The Lorentz factor explains time dilation and space contraction, two phenomena that result due to the constancy of the speed of light. Time dilation is observed when the time measured on the clock of a moving observer, or an observer at rest relative to an event, is slower when measured by a stationary observer. Space contraction is the shortening of distance measured by a moving observer when measured by a stationary observer.[2] The Lorentz factor is a key component in the Lorentz transformations, which convert space and time coordinates from one reference frame to another under the postulates of special relativity.[3]

Rapidity is introduced and is related to velocity via the hyperbolic tangent function.[4] The relationship is as follows: $v = c\,tanh\,(\phi)$. This simplifies calculations, as rapidity is added linearly under the hyperbolic tangent function. In this paper, we attempt to generalize the velocity addition formula introduced by Einstein for any number of compositions, due to its increasing complexity with additional non-linear sums.

To achieve this generalization, we use elementary symmetric polynomials, utilizing the fact that hyperbolic and trigonometric function compositions are combinatorial by nature and to show that such compositions are foundational structures to symmetric polynomials.[5] Elementary symmetric polynomials help simplify notation as seen later in this paper. Previously, Cook's work in 2023 has shown that there indeed exists a connection between elementary symmetric polynomials and the hyperbolic tangent.[6] However, we will present a new method to prove the connection. In doing so, we will also derive a new formulation of this relationship.

The results found in this paper are important because a closed-form generalization of relativistic velocity addition simplifies calculations when adding three or more velocities; in addition, it further validates Einstein's theory of special relativity by proving that, for any number of relativistic velocity additions, it does not exceed the speed of light. This obeys the postulates of special relativity. Furthermore, this paper bridges a gap between abstract algebra, discrete mathematics, and special relativity, because it is found that the generalizations of relativistic velocity addition, the Lorentz factor, and the Lorentz transformations for one dimension are combinatorial and are all describable with the use of elementary symmetric polynomials. Moreover, this paper provides a closed-form formulation of the generalization of the composition of the hyperbolic tangent function, explicitly displaying elementary symmetric polynomials in its structure; this is a key distinction to other generalizations of the hyperbolic tangent

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[1] Albert Einstein, "On the electrodynamics of moving bodies," *Annalen der Physik* 17, no. 10 (1905): 891-921: 12.

[2] Albert Einstein, "On the electrodynamics of moving bodies," 10.

[3] Albert Einstein, "On the electrodynamics of moving bodies," 7.

[4] Vladimir Varicak, "Application of Lobachevskian geometry in the theory of relativity," *Phys. Z* 11 (1910): 93-96: 2.

[5] Adel F. Antippa, "The combinatorial structure of trigonometry," *International Journal of Mathematics and Mathematical Sciences* 2003, no. 8 (2003): 477

[6] John Cook, "Tanh and Elementary Symmetric Polynomials," John D. Cook Consulting, October 7, 2023, https://www.johndcook.com/blog/2023/10/07/tanh-sum-proof

function compositions. Finally, the result that the Lorentz factor converges suggests that the amount of time dilation and space contraction that occurs is limited for certain sequences of velocities.

PRELIMINARIES

1. Lorentz Factor

Let v be the relative velocity between two coordinate systems and c be the speed of light in vacuum. Then, the Lorentz factor, as mentioned before arises due to the effects of time dilation and space contraction. The Lorentz factor is given as follows:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

2. Lorentz Transformations

Lorentz transformations convert the spatial coordinates, x, and the temporal coordinates, t, of one reference frame into another reference frame (coordinates are x' and t'). These are given as:

$$x' = \gamma(x - vt) \text{ and } t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

3. Hyperbolic Tangent Composition and Relativistic Velocity Addition

The hyperbolic tangent function composition is a non-linear addition. Let $\phi_1, \phi_2 \in \mathbb{R}$. For two compositions, the formula reads:

$$tanh(\phi_1 + \phi_2) = \frac{tanh(\phi_1)+tanh(\phi_2)}{1+tanh(\phi_1)tanh(\phi_2)}$$

When converting into velocity, using $v = ctanh(\phi)$, the relativistic velocity addition is non-linear. Non-linear addition of velocities is not a direct addition of the velocities. Instead, it involves a more complex operation such as multiplication within the addition. The resultant relativistic velocity is as follows for velocities v_1, v_2, and the speed of light, c:

$$v_1 \oplus v_2 = \frac{v_1+v_2}{1+(v_1v_2)/c^2}$$

Velocity in physics is a vector quantity that describes the magnitude and direction of an object's speed. Linear velocity addition refers to velocities added directly. It does not take into account any relativistic effects. The resultant velocity is given by $\omega_i = \sum_{j=1}^{i} v_j$.

4. Elementary Symmetric Polynomials

For $r \geq 0$, the elementary symmetric polynomial e_r is the sum of all products of r distinct x_i, where $i \in \mathbb{N}$ and $0 \leq r \leq i$, such that:

$$e_0 = 1 \, and \, e_r = \sum_{1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \ldots < \alpha_r \leq i} x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} \ldots x_{\alpha_r}$$

For the elementary symmetric polynomials:[7]

$$\sum_{r=0}^{i} e_r t^r = \prod_{j=1}^{i} (1 + x_j t)$$

We introduce two identities by substituting t as 1 and -1 respectively:

$$\sum_{r=0}^{i} e_r = \prod_{j=1}^{i} (1 + x_j)$$

$$\sum_{r=0}^{i} (-1)^r e_r = \prod_{j=1}^{i} (1 - x_j)$$

5. Convergence Conditions

Theorem 1: The convergence of infinite products.[8] For $a_n > 0$ and $n \geq 1$, the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ converges iff $\sum_{j=1}^{\infty} a_n$ converges.

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[7] Ian Grant Macdonald, *Symmetric functions and Hall polynomials* (Oxford: Oxford University Press, 1998), 19.

[8] "Notes on the Riemann Zeta Function," class notes for MATH 324: *Elementary Number Theory*, University of Alberta, 2012. https://www.math.ualberta.ca/~isaac/math324/s12/zeta.pdf

Theorem 2: If $a_n \geq 0$ for all $n \geq 1$, the infinite product $\prod_{n=1}^{\infty}\left(1 - a_n\right)$ converges iff $\sum_{n=1}^{\infty} a_n$ converges. To view both proofs, see [Notes on the Riemann Zeta Function, pp. 4-5].

MAIN RESULTS

Theorem 1

For all $\phi_1, \phi_2, \dots, \phi_i \in \mathbb{R}$ and $i \geq 1$, the result of adding i rapidities under the hyperbolic function tangent function is as follows:

$$tanh\left(\sum_{j=1}^{i} \phi_j\right) = \frac{\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} e_{2r+1}(tanh(\phi_1), tanh(\phi_2), \dots, tanh(\phi_i))}{\sum_{r=0}^{\lfloor i/2 \rfloor} e_{2r}(tanh(\phi_1), tanh(\phi_2), \dots, tanh(\phi_i))}$$

For the sake of simplicity, we denote:

$$O_i = \sum_{r=0}^{\lfloor (i-1)/2 \rfloor} e_{2r+1}(tanh(\phi_1), tanh(\phi_2), \dots, tanh(\phi_i))$$

and

$$E_i = \sum_{r=0}^{\lfloor i/2 \rfloor} e_{2r}(tanh(\phi_1), tanh(\phi_2), \dots, tanh(\phi_i))$$

Proof of Theorem 1

As mentioned before, we use mathematical induction to show that elementary symmetric polynomials arise in hyperbolic tangent function composition.
Let $\phi_1, \phi_2, \phi_3, \dots, \phi_i \in \mathbb{R}$, and let $x_i = tanh\left(\phi_i\right)$.

For $i = 1$

$$tanh\left(\sum_{j=1}^{1} \phi_j\right) = \frac{\sum_{r=0}^{\lfloor (1-1)/2 \rfloor} e_{2r+1}(tanh(\phi_1))}{\sum_{r=0}^{\lfloor 1/2 \rfloor} e_{2r}(tanh(\phi_1))} = \frac{e_1(tanh(\phi_1))}{e_0(tanh(\phi_1))} = tanh(\phi_1)$$

Assume $i = k$ is true. That is, we assume:

$$tanh\left(\sum_{j=1}^{k} \phi_j\right) = \frac{\sum_{r=0}^{\lfloor (k-1)/2 \rfloor} e_{2r+1}(tanh(\phi_1), tanh(\phi_2), \dots, tanh(\phi_k))}{\sum_{r=0}^{\lfloor k/2 \rfloor} e_{2r}(tanh(\phi_1), tanh(\phi_2), \dots, tanh(\phi_k))} = \frac{O_k}{E_k}$$

Here, we introduce floor functions to unify the two cases.

Case 1: $i = k$ is odd.

In the numerator, O_k is as follows:

$$e_1 + e_3 + \dots + e_k = \sum_{r=0}^{(k-1)/2} e_{2r+1}$$

The denominator is:

$$e_0 + e_2 + \dots + e_{k-1} = \sum_{r=0}^{(k-1)/2} e_{2r}$$

Case 2: $i = k$ is even

In the numerator, O_k, is as follows:

$$e_1 + e_3 + \dots + e_{k-1} = \sum_{r=0}^{(k-2)/2} e_{2r+1}$$

The denominator is:

$$e_0 + e_2 + \dots + e_k = \sum_{r=0}^{k/2} e_{2r}$$

Whether k is even and odd, the highest odd-order elementary symmetric polynomial is $2r + 1 \leq k$. In addition, $2r + 1$ can only be a whole number. Hence we unify both cases for the numerator under the floor function $\lfloor \frac{k-1}{2} \rfloor$. The highest even-order elementary symmetric polynomial is $2r \leq k$. Hence, we unify both cases for the denominator under the floor function $\lfloor \frac{k}{2} \rfloor$. The proof is complete if the following is shown to be valid for $i = k + 1$. That is, we want to show:

$$tanh\left(\sum_{j=1}^{k+1} \phi_j\right) = \frac{\sum_{r=0}^{\lfloor k/2 \rfloor} e_{2r+1}(tanh(\phi_1), \dots, tanh(\phi_{k+1}))}{\sum_{r=0}^{\lfloor (k+1)/2 \rfloor} e_{2r}(tanh(\phi_1), \dots, tanh(\phi_{k+1}))} = \frac{O_{k+1}}{E_{k+1}}$$

$$tanh\left(\sum_{j=1}^{k+1} \phi_j\right) = tanh\left(\sum_{j=1}^{k} \phi_j + \phi_{k+1}\right) = \frac{\frac{O_k}{E_k} + tanh(\phi_{k+1})}{1+\left(\frac{O_k}{E_k}\right)(tanh(\phi_{k+1}))} = \frac{O_k + E_k tanh(\phi_{k+1})}{E_k + O_k tanh(\phi_{k+1})}$$

Letting $x_i = tanh(\phi_i)$ and a key classical theorem which is the following:[9]

$$e_m(x_1, x_2, \dots, x_k) + e_{m-1}(x_1, x_2, \dots, x_k) \cdot x_{k+1} = e_m(x_1, x_2, \dots, x_{k+1})$$

$$= (tanh(\phi_1), tanh(\phi_2), \dots, tanh(\phi_k)) + e_{m-1}(tanh(\phi_1), tanh(\phi_2), \dots, tanh(\phi_k)) \cdot tanh(\phi_{k+1})$$

$$= e_m(tanh(\phi_1), tanh(\phi_2), \dots, tanh(\phi_{k+1}))$$

Examining the numerator:

$$\sum_{r=0}^{\lfloor (k-1)/2) \rfloor} e_{2r+1}(tanh(\phi_1), \dots, tanh(\phi_k)) + \sum_{r=0}^{\lfloor k/2 \rfloor} e_{2r}(tanh(\phi_1), \dots, tanh(\phi_k)) \cdot tanh(\phi_{k+1})$$

By the theorem previously introduced the expression therefore is equivalent to:

$$\sum_{r=0}^{\lfloor k/2 \rfloor} e_{2r+1}(tanh(\phi_1), \dots, tanh(\phi_{k+1})) = O_{k+1}$$

For the denominator, the expression is as follows:

$$\sum_{r=0}^{\lfloor k/2 \rfloor} e_{2r}(tanh(\phi_1), \dots, tanh(\phi_k)) + \sum_{r=0}^{\lfloor (k-1)/2 \rfloor} e_{2r+1}(tanh(\phi_1), \dots, tanh(\phi_k)) \cdot tanh(\phi_{k+1}))$$

Again, by the theorem the expression then equals:

$$\sum_{r=0}^{\lfloor (k+1)/2 \rfloor} e_{2r}(tanh(\phi_1), \dots, tanh(\phi_{k+1})) = E_{k+1}$$

Hence, the formula is proven by mathematical induction.

$$Q.E.D.$$

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[9] Maryam Salem Alatawi, "On the Elementary Symmetric Polynomials and the Zeroes of Legendre Polynomials," *Journal of Mathematics* (2022): 2, https://doi.org/10.1155/2022/4139728.

Theorem 2

Using the relationship between velocity and rapidity, we obtain the generalization of relativistic velocity addition in one dimension after any number of successive collinear additions.
We denote, for simplicity, the total addition of velocities after i additions as ω_i. For all $i \geq 1$, the collinear addition of $v_1, v_2, v_3, \ldots, v_i$ under the relativistic addition operator, where c is the constant speed of light is given as follows:

$$\omega_i = v_1 \oplus v_2 \oplus v_3 \oplus \ldots \oplus v_i = \frac{\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} 1/c^{2r} e_{2r+1}(v_1, v_2, \ldots, v_i)}{\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} e_{2r}(v_1, v_2, \ldots, v_i)}$$

Proof of Theorem 2

We have now established the hyperbolic tangent composition for an arbitrary i number of additions. After converting the rapidities into velocities using the relationship, $v = ctanh(\phi)$, the formula is as follows:

$$\omega_i = ctanh\left(\sum_{j=1}^{i} \phi_j\right) = v_1 \otimes v_2 \otimes \ldots \otimes v_i = \frac{c \times \sum_{r=0}^{\lfloor (i-1)/2 \rfloor} e_{2r+1}\left(\frac{v_1}{c}, \frac{v_2}{c}, \ldots, \frac{v_i}{c}\right)}{\sum_{r=0}^{\lfloor i/2 \rfloor} e_{2r}\left(\frac{v_1}{c}, \frac{v_2}{c}, \ldots, \frac{v_i}{c}\right)}$$

$$= \frac{c \times \sum_{r=0}^{\lfloor (i-1)/2 \rfloor} 1/c^{2r+1} e_{2r+1}\left(v_1, v_2, \ldots, v_i\right)}{\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} e_{2r}\left(v_1, v_2, \ldots, v_i\right)} = \frac{\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} 1/c^{2r} e_{2r+1}\left(v_1, v_2, \ldots, v_i\right)}{\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} e_{2r}\left(v_1, v_2, \ldots, v_i\right)}$$

To prove that the relativistic addition of the velocities do not exceed c for any arbitrary composition, consider the scenario where all $v_i = c$. That is, consider:

$$= \frac{c \times \sum_{r=0}^{\lfloor (i-1)/2 \rfloor} 1/c^{2r+1} e_{2r+1}\left(v_1, v_2, \ldots, v_i\right)}{\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} e_{2r}\left(v_1, v_2, \ldots, v_i\right)} = \frac{\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} 1/c^{2r} e_{2r+1}\left(v_1, v_2, \ldots, v_i\right)}{\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} e_{2r}\left(v_1, v_2, \ldots, v_i\right)}$$

The coefficients of the terms in the numerator are all odd combinations, and all the combinations in the denominator are even. Using the binomial expansion, we can show:

$$(1+1)^i = \sum_{k=0}^{i} C(i,k) = 2^i \; and \; (1-1)^i = \sum_{k=0}^{i} (-1)^r C(i,k) = 0$$

Combining the odd-order combinations in the numerator with the even-order combinations in the denominator, the following is shown:

$$\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} C(i, 2r+1) + \sum_{r=0}^{\lfloor i/2 \rfloor} C(i, 2r) = \sum_{n=0}^{i} C(i,n) = 2^i$$

It is implied that:

$$\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} C(i, 2r+1) = \sum_{r=0}^{\lfloor i/2 \rfloor} C(i, 2r) = \frac{2^i}{2} = 2^{i-1}$$

since all even-order combinations are positive, while all the odd-order are negative in the expansion of $(1-1)^i$. Therefore, in the original equation, when all velocities are at c, we prove:

$$\frac{\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} C(i, 2r+1)\cdot c}{\sum_{r=0}^{\lfloor i/2 \rfloor} C(i, 2r)} = \frac{2^{i-1}c}{2^{i-1}} = c$$

We observe that the established formula for the generalization of relativistic velocity addition remains within the postulates of special relativity by not exceeding c.

Corollary 1

We also derive the generalization of the Lorentz factor for collinear boosts. For all $i \geq 1$, the total effect of the Lorentz factor after i number of collinear boosts with relative velocities $v_1, v_2, v_3, \dots, v_i$, where c is the constant speed of light is given as:

$$\gamma = \prod_{j=1}^{i} \gamma_j \times \left(\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} e_{2r}(v_1, v_2, \dots, v_i) \right) \text{ where } \gamma_j = \frac{1}{\sqrt{1-v_j^2/c^2}}$$

Proof of Corollary 1

The Lorentz factor is a function of velocity. Suppose the resultant velocity is the sum of all the relativistic additions of v_j, which we denote as ω_i. Then the Lorentz factor is as follows:

$$\gamma = \frac{1}{\sqrt{1-\omega_i^2/c^2}}$$

Using the relationship between relativistic velocity and hyperbolic tangent again:

$$\omega_i = c\tanh\left(\sum_{j=1}^{i} \phi_j\right) = \frac{c\cdot O_i}{E_i}$$

The Lorentz factor is as given:

$$\gamma = \frac{1}{\sqrt{1-O_i^2/E_i^2}} = 1/\left(\frac{\sqrt{E_i^2-O_i^2}}{E_i}\right) = \frac{E_i}{\sqrt{E_i^2-O_i^2}} = \frac{E_i}{\sqrt{(E_i-O_i)(E_i+O_i)}}$$

The expressions $E_i - O_i$ and $E_i + O_i$ can be presented as

$$E_i + O_i = \sum_{r=0}^{\lfloor i/2\rfloor} e_{2r} + \sum_{r=0}^{\lfloor (i-1)/2\rfloor} e_{2r+1} = \sum_{k=0}^{i} e_k = \prod_{j=1}^{i} (1 + \tanh(\phi_j))$$

$$E_i - O_i = \sum_{r=0}^{\lfloor i/2\rfloor} e_{2r} - \sum_{r=0}^{\lfloor (i-1)/2\rfloor} e_{2r+1} = \sum_{k=0}^{i} (-1)^k e_k = \prod_{j=1}^{i} (1 - \tanh(\phi_j))$$

Hence, after substituting these representations, the Lorentz factor is

$$\gamma = \frac{E_i}{\sqrt{\left(\prod_{j=1}^{i}(1-\tanh(\phi_j)\right)\left(\prod_{j=1}^{i}(1+\tanh(\phi_j)\right)}} = \frac{E_i}{\sqrt{\left(\prod_{j=1}^{i}(1-\tanh^2(\phi_j)\right)}} = \prod_{j=1}^{i} \frac{E_i}{\sqrt{1-\tanh^2(\phi_j)}}.$$

In terms of velocities, the established generalization of the Lorentz factor for any number of collinear boosts:

$$\gamma = \left(\prod_{j=1}^{i} \frac{1}{\sqrt{1-v_j^2/c}}\right)\left(\sum_{r=0}^{\lfloor i/2\rfloor} 1/c^{2r} e_{2r}(v_1, v_2, \dots, v_i)\right).$$

When $i = 1$,

$$\gamma = \left(\prod_{j=1}^{1} \frac{1}{\sqrt{1-v_j^2/c}}\right)\left(\sum_{r=0}^{\lfloor 1/2 \rfloor} 1/c^{2r} e_{2r}(v_1)\right) = \frac{1}{\sqrt{1-v_1^2/c}}.$$

When $i = 2$,

$$\gamma = \left(\prod_{j=1}^{2} \frac{1}{\sqrt{1-v_j^2/c}}\right)\left(\sum_{r=0}^{\lfloor 1 \rfloor} 1/c^{2r} e_{2r}(v_1, v_2)\right) = \left(\frac{1}{\sqrt{1-v_1^2/c}\sqrt{1-v_2^2/c}}\right)\left(e_0(v_1, v_2)/c^0 + e_2(v_1, v_2)/c^2\right)$$

$$= \left(\frac{1}{\sqrt{1-v_1^2/c}\sqrt{1-v_2^2/c}}\right)\left(1 + \frac{v_1 v_2}{c^2}\right).$$

Corollary 2

The interval of velocities when analyzing the convergence of the Lorentz factor as the number of collinear boosts approaches infinity is: $0 < v_j < c$ and hence both v_j^2/c and v_j^2/c^2 are within the interval (0, 1). For all $j \geq 1$ and $0 < v_j/c < 1$, the generalization of the Lorentz factor as $i \to \infty$ converges iff $\sum_{j=1}^{\infty} v_j/c < \infty$ and $\sum_{j=1}^{\infty} v_j^2/c^2 < \infty$.

Proof of Corollary 2

To analyze the convergence of the Lorentz Factor, we rearrange the following expression:

$$\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} e_{2r}(v_1, v_2, \dots, v_i) = \frac{1}{2}\prod_{j=1}^{i}\left(1 + v_j/c\right) + \frac{1}{2}\prod_{j=1}^{i}\left(1 - v_j/c\right),$$

using the identities:

$$\prod_{j=1}^{i}\left(1 + X_j\right) = \sum_{r=0}^{i} e_r\left(X_1, X_2, \dots, X_i\right) \text{ and } \prod_{j=1}^{i}\left(1 - X_j\right) = \sum_{r=0}^{i} (-1)^r e_r\left(X_1, X_2, \dots, X_i\right).$$

Hence, as the number of velocity additions approach infinity, the Lorentz factor then may be expressed as:

$$\gamma = \prod_{j=1}^{\infty}\left(1 - v_j^2/c^2\right)^{-1/2}\left(\frac{1}{2}\prod_{j=1}^{\infty}(1 + v_j/c) + \frac{1}{2}\prod_{j=1}^{\infty}(1 - v_j/c)\right)$$

For the sake of simplicity, in this scenario, we will study the convergence of the Lorentz factor for which all the velocities are collinear and in the same direction. That is for any $j,\ 0 < v = v_j < c$. The Lorentz factor converges if all three infinite products in the Lorentz factor converge. We know from classical algebraic theorems that for every $v/c > 0$, both products, $\prod(1 + v/c)$ *and* $\prod(1 - v/c)$, converge to a non-zero number iff $\sum v/c$ converges. Note that both v/c and v^2/c^2 are within the interval [0,1) for all v.

Theorem: $\prod(1 - v^2/c^2)^{-1/2}$ converges to a non-zero number iff $\sum v^2/c^2 < \infty$.

Proof

We know that if $v_j^2/c^2 \geq 0$ for all $j \geq 1$, then $\prod_{j=1}^{\infty}\left(1 - v_j^2/c^2\right)$ converges to a non-zero number iff $\sum_{j=1}^{\infty} v_j^2/c^2$ converges. We note that v^2/c^2 can only be in the interval [0,1) under the postulates of special relativity. Therefore, the infinite product can only converge to a positive integer. Hence, if $\prod_{j=1}^{\infty}\left(1 - v_j^2/c^2\right) = K$ and if $K > 0$, then $1/\sqrt{K}$ exists since $f(x) = 1/\sqrt{x}$ is continuous for all $x > 0$.

As a result, if 0≤ $v_j^2/c^2 < 1$ for all $j \geq 1$, then $\prod_{j=1}^{\infty}\left(1 - v_j^2/c^2\right)^{-1/2}$ converges iff $\sum_{j=1}^{\infty} v_j^2/c^2$ converges.

$Q.E.D.$

Finally, we conclude that the generalization of the Lorentz factor converges for an infinite number of collinear additions, in the same direction, if the following conditions are satisfied: $\sum_{j=1}^{\infty} v_j/c$ and $\sum_{j=1}^{\infty} v_j^2/c^2$ both converge. Taking an example, let $\sum_{j=1}^{\infty} v_j/c = \sum_{j=1}^{\infty} 1/j$, and therefore, $\sum_{j=1}^{\infty} v_j^2/c^2 = \sum_{j=1}^{\infty} 1/j^2$. We see that, by the p-series, the first condition diverges and the second condition converges. Since both conditions were not satisfied, the Lorentz factor diverges. However, consider the case when the velocities are given as: $\sum_{j=1}^{\infty} v_j/c = \sum_{j=1}^{\infty} 1/j^3$ and $\sum_{j=1}^{\infty} v_j^2/c^2 = 1/j^6$. By the p-series, both infinite series converge, and, hence, all three infinite products converge. As a result, the Lorentz factor converges.

Corollary 3

Finally, both the spatial and temporal generalizations of the Lorentz transformations are developed. For $i \geq 1$, the spatial coordinate, $x^{(i)}$, of reference frame S^i in terms of the spatial coordinate, x, of stationary reference frame S^0 where v_i is the relative velocity between reference frames S^i and S^{i-1}, is given as:

$$x^{(i)} = \prod_{j=1}^{i} \left(1 - v_j^2/c^2\right)^{-1/2} \left(\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} \left(v_1,\ v_2,\ \dots\ ,\ v_i\right) x \ - \ \sum_{r=0}^{\lfloor (i-1)/2 \rfloor} 1/c^{2r} \left(v_1,\ v_2,\ \dots\ ,\ v_i\right) t \right)$$

For $i \geq 1$, the temporal coordinate, $t^{(i)}$, of reference frame S^i in terms of the temporal coordinate, t, of stationary reference frame S^0 where v_i is the relative velocity between reference frames S^i and S^{i-1}, is given as:

$$t^{(i)} = \prod_{j=1}^{i} \left(1 - v_j^2/c^2\right)^{-1/2} \left(\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} \left(v_1,\ v_2,\ \dots\ ,\ v_i\right) t \ - \sum_{r=0}^{\lfloor (i-1) \rfloor} 1/c^{2+2r} \left(v_1,\ v_2,\ \dots\ ,\ v_i\right) x \right).$$

Proof of Corollary 3

In this following section, both the spatial and the temporal components of the Lorentz transformations are addressed. Starting with the spatial transformation, suppose that $x^{(i)}$ represents the spatial coordinates after the ith transformation and x represents the stationary reference frame. The $x^{(i)}$ notation is a replacement, as adding many apostrophes would be difficult to do. For example, $x^{(5)}$ is equivalent to x'''''; $x^{(0)}$ represents x which is the stationary coordinate system. Also, let the sequence $v_1,\ v_2,\ \dots\ ,\ v_i$ be the relative velocities between the inertial coordinate systems, and let ω_i be the resultant velocity. Then, the spatial Lorentz transformation is given as:

$$x^{(i)} = \frac{1}{\sqrt{1-\omega_i^2/c^2}} \left(x - \omega_i t\right)$$

$$= \prod_{j=1}^{i} \left(1 - v_j^2/c^2\right)^{-1/2} \left(\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} \left(v_1,\ v_2,\ \dots\ ,\ v_i\right) \right) \left(x - \frac{\sum_{r=0}^{\lfloor (i-1)/2 \rfloor} 1/c^{2r} \left(v_1, v_2, \dots , v_i\right)}{\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} \left(v_1, v_2, \dots , v_i\right)} t \right)$$

$$= \prod_{j=1}^{i} \left(1 - v_j^2/c^2\right)^{-1/2} \left(\sum_{r=0}^{\lfloor i/2 \rfloor} 1/c^{2r} \left(v_1,\ v_2,\ \dots\ ,\ v_i\right) x \ - \ \sum_{r=0}^{\lfloor (i-1)/2 \rfloor} 1/c^{2r} \left(v_1,\ v_2,\ \dots\ ,\ v_i\right) t \right)$$

In addition, suppose that, after i number of collinear Lorentz transformations in the same direction, the temporal component, $t^{(i)}$ is given as:

$$t^{(i)} = \frac{1}{\sqrt{1-\omega_i^2/c^2}}\left(t - x\omega_i/c^2\right)$$

$$= \prod_{j=1}^{i}\left(1 - v_j^2/c^2\right)^{-1/2}\left(\sum_{r=0}^{\lfloor i/2\rfloor} 1/c^{2r}\left(v_1, v_2, \dots, v_i\right)\right)\left(t - x\left(\frac{\sum_{r=0}^{\lfloor (i-1)/2\rfloor} 1/c^{2r}\left(v_1, v_2, \dots, v_i\right)}{\sum_{r=0}^{\lfloor i\rfloor} c^{2-2r}\left(v_1, v_2, \dots, v_i\right)}\right)\right)$$

$$= \prod_{j=1}^{i}\left(1 - v_j^2/c^2\right)^{-1/2}\left(\sum_{r=0}^{\lfloor i/2\rfloor} 1/c^{2r}\left(v_1, v_2, \dots, v_i\right)t - \sum_{r=0}^{\lfloor (i-1)\rfloor} 1/c^{2+2r}\left(v_1, v_2, \dots, v_i\right)x\right).$$

This generalization of the Lorentz transformations in 1 dimension allow us to convert both the spatial and temporal components of the ith coordinate system in terms of the "stationary" one, given the relative velocities of every coordinate system between them. In order for these generalizations to be valid, the coordinate systems are required to be travelling at a constant velocity relative to one another; in addition, they must be travelling collinearly or on a parallel straight line to each other.

DISCUSSION

The rise of elementary symmetric polynomials in the composition of the hyperbolic tangent function is due to the non-linear addition, involving multiplication. When this operation is repeated several times, it fits precisely what the elementary symmetric polynomials describe: for $r \geq 0,$ the sum of the products of r distinct $tanh\left(\phi_i\right)$. The numerator consists of all odd-order elementary symmetric polynomials and the denominator consists of all even-order elementary symmetric polynomials. This corresponds to the identity: $tanh(\phi) = sinh(\phi)/cosh(\phi)$, where $sinh(\phi)$ is an *odd* function, and $cosh(\phi)$ is an *even* function.

Naturally, after it was established that the combinatorial structure of the hyperbolic tangent was describable by elementary symmetric polynomials, using the conversion between velocities and rapidities, it was found that relativistic velocity addition also gave rise to elementary symmetric polynomials. The velocity structure does not exceed the speed of light because when all the velocities were the speed of light, the maximum sum in the numerator was $2^{i-1} \cdot c$ and the maximum sum of all velocities in the denominator was 2^{i-1}. The two identical values cancel out, leaving only the speed of light, supporting the validity of the derived structure, because it follows the postulates of special relativity.

Since the Lorentz factor is a function of velocity, it was also found that it was describable by elementary symmetric polynomials. After a substantial amount of algebraic manipulation, the Lorentz factor was generalized. Theorems on the convergence of infinite products allowed us to analyze the conditions sufficient for the Lorentz factor to converge. The convergence requires, as seen in the theorems, a specific sequence of velocities, suggesting that, of all the possible velocity sequences and the fact that the sequence must satisfy two conditions, it is more likely for the Lorentz factor to diverge than to converge in a given scenario where an infinite number of relativistic velocities are added.

CONCLUSION

In this paper, we have addressed and solved the cumbersome nature of collinear relativistic velocity additions, by utilizing elementary symmetric polynomials and floor functions to describe the combinatorial structure that arises after many non-linear additions.

We proved this through mathematical induction and the hyperbolic tangent function, providing a new perspective on the rise of elementary symmetric polynomials in hyperbolic functions. Subsequently, we demonstrated and proved that the established formulation was in accordance with the postulate of special relativity that no velocity can exceed the speed of light. We proved this using key binomial theorems and implications. We then considered the Lorentz factor and developed a generalization of it valid for any number of collinear boosts. In addition, we used several algebraic theorems to study the conditions by which the Lorentz factor converges for an infinite number of collinear boosts in a purely abstract scenario. Finally, we investigated the generalization of any number of Lorentz transformations, which can convert any spatial or temporal components of coordinate systems given the relative velocity between reference frames.

This whole paper is applied only for one dimensional collinear boosts in the same direction. Future work on this paper could involve extending this combinatorial framework to higher dimensions and expanding this framework to other hyperbolic and trigonometric functions. One could also investigate possible interpretations of the meaning of a diverging or converging Lorentz factor. Furthermore, future work can be using this framework to develop generalized formulations of relativistic kinetic energy and momentum.

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