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Traffic Growth and Clearance at a Signalised On-Ramp Merge: An LWR Analysis

Brianna Hou
brianna.lanjing.hou@gmail.com

ABSTRACT

The coincidence of a traffic light and a merging on-ramp is a common feature of urban and highway road design. The single-lane signalised road is a standard example in the study of hyperbolic conservation laws, but the interaction between these two aspects is comparatively underexplored. This paper models this using the Lighthill-Whitham-Richards framework, treating traffic as a continuous fluid whose density evolves according to a first-order hyperbolic partial differential equation. Using the method of characteristics, the Rankine-Hugoniot shock condition, and a rarefaction fan for the green-phase release, closed-form expressions are derived for the maximum traffic jam length and clearance time as functions of traffic light timing and ramp inflow rate.

The model is compared against a cell-transmission (CTM) numerical solution of the underlying conservation law. This comparison confirms the qualitative mechanisms captured by the model, jam growth during red, a two-wave clearance structure during green, and the role of ramp inflow in tightening the stability margin, and agrees quantitatively: for the tested cases the analytic jam length and clearance time lie within about 2% of the simulation. The stability condition is derived from cycle-level vehicle conservation, and the original test parameters are shown to be oversaturated, which the simulation correctly reflects. The paper concludes with an outline of further extensions for continued research.

Keywords: *LWR model; kinematic waves; rarefaction fan; signalised intersection; on-ramp merge; cell transmission model*

INTRODUCTION

Traffic flow and traffic congestion are often treated as a purely engineering problem, something to be measured and simulated. However, it also obeys precise mathematical laws. If traffic is treated as a continuous fluid rather than a collection of individual vehicles, the flow of cars along a road can be described by a partial differential equation. Sudden jumps in density represent real-world traffic jams that are called **shocks** and are similar to mathematics describing shock waves in compressible fluid dynamics.

This paper will address two different scenarios. The first is a straight highway with one traffic light. The second is a more underexplored situation with a single point on a road where a traffic light and a merging

on-ramp coincide. Both traffic lights and on-ramps are studied individually in terms of traffic flow, but the case where they are at the same location, which affects each other, has received less attention.

Related Work: The kinematic-wave analysis of queues at signalised links was developed by Stephanopoulos, Michalopoulos and Stephanopoulos (1979) and Michalopoulos, Stephanopoulos, and Pisharody (1981), and Newell's simplified kinematic wave theory (1993) gives a compact treatment of queues at bottlenecks. A stop line that restricts the flow through a point is a conservation law with a flux constraint, whose Riemann problem was studied by Colombo and Goatin (2007). Junctions on networks are handled by Riemann solvers (Holden and Risebro, 1995; Coclite, Garavello and Piccoli, 2005; Garavello and Piccoli, 2006) and, in discrete form, by the demand-supply formalism of the cell-transmission model (Daganzo, 1994, 1995; Lebacque, 1996). Kolbe (2024) studies coupling conditions that preserve the ratio of incoming flows at networked LWR junctions. A metered on-ramp is itself a signal at a merge, and feedback strategies for it, such as ALINEA (Papageorgiou, Hadj-Salem and Blosseville, 1991), are well established in the control literature. The single-link results are classical and are not claimed as new here. What this paper adds is a closed-form treatment, within one LWR framework, of a fixed-time signal whose stop-line capacity is shared between a spatial main-road queue and a ramp queue, including the interaction of the release wave with the back of the queue and a cycle-level stability condition written in terms of the ramp inflow.

The main questions addressed in this paper are, given a traffic light with a fixed cycle length and red/green split, how long does a traffic jam grow and how long does it take to clear once the light turns green. This paper derives the behavior using the Lighthill-Whitham-Richards (LWR) model of traffic flow, verifying the resulting formulas against a direct numerical solution of the conservation law.

Section 2 of this paper reviews the LWR model and other mathematics used in the rest of the paper. Section 3 sets up the specific merged model. Section 4 derives expressions for the traffic growth and clearance, section 5 analyzes stability and special cases, and section 6 compares the analytical results with a numerical solution. Section 7 concludes and outlines future work, including a comparison with real data.

The Lighthill-Whitham-Richards Model

The Conservation Law

Considering a single-lane road parametrized by position x, let $\rho(x,t)$ denote the density of cars (vehicles per unit length) at position x and time t. The fundamental modeling assumption of the Lighthill-Whitham-Richards model is that, over a short enough length scale, traffic can be treated as a continuous fluid. No cars are created or destroyed, so the only way the number of cars in a segment of road can change is by cars crossing the segment's boundaries (Lighthill and Whitham, 1955).

Fix a segment $[x,\ x + \Delta x]$. The number of cars in this segment at time t is approximately $\rho(x,t)\Delta x$. Let $q(x,\ t)$ denote the **flux**, which is the rate at which cars pass position x at time t in vehicles per unit time.

Conservation of cars requires that the rate of change of the number of cars in the segment equals the flux in at x minus the flux out at $x + \Delta x$

(1) $$\frac{d}{dt}[\rho(x,\ t)\Delta x] \approx\ q(x,\ t) - q(x + \Delta x,\ t)$$

Dividing by Δx and letting $\Delta x \to 0$ gives the continuity equation

(2) $$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

This equation is currently written as two unknown functions ρ and q. The LWR model then makes a second assumption. The flux depends on density alone where

(3) $$q(x,\ t) = \rho(x,\ t)v(\rho(x,\ t))$$

where $v(\rho)$ is a chosen velocity-density relationship. This assumption is that drivers instantaneously adjust their speeds based only on the local density around them with no dependence on acceleration, anticipation, or reaction time (Richards, 1956).

The Velocity-Density Relation

The function $v(\rho)$ should be a decreasing function of ρ where drivers travel near the speed v_{max} when the density is low and slow down to almost zero as density approaches maximum ρ_{max}. This is a traffic jam where $v \to 0$. The function

(4) $$v(\rho) = v_{max}(1 - \frac{\rho}{\rho_{max}})$$

is the simplest linear relation satisfying both maximum and minimum boundary conditions (Greenshields, 1935). This gives a flux function

(5) $$q(\rho) = \rho v(\rho) = v_{max}\rho(1 - \frac{\rho}{\rho_{max}})$$

This is a downward-opening parabola in ρ and is 0 both at ρ = 0 (no cars, no flux) and ρ = ρ_{max} (almost no motion → no flux). It is maximized at $\rho = \frac{\rho_{max}}{2}$ where flux is at

(6) $$q_{max} = \frac{v_{max}\rho_{max}}{4}$$

(Greenshields, 1935). For every flux below q_{max}, there are two densities with that flux, a free-flow density below $\frac{\rho_{max}}{2}$ and a congested density above it.

Method of Characteristics

Equation (2), with $q = q(\rho)$, can be written using the chain rule as

(7) $$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0$$

This is a first-order partial differential equation and can be solved using the method of characteristics. Along any curve *x(t)* satisfying

(8) $$\frac{dx}{dt} = q'(\rho)$$

using the chain rule,

(9) $$\frac{d}{dt}\rho(x(t),\, t) = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0$$

ρ is constant along each characteristic curve, and since ρ is constant along it, so is *q(ρ)*. This means each characteristic curve is a straight line so $x(t) = x_0 + q(\rho_0)t$ where ρ_0 is the initial density at the characteristic's starting point x_0.

So if initial density is increasing in *x* over some region, then since $q(\rho) = v_{max}(1 - \frac{2\rho}{\rho_{max}})$ is a decreasing function of ρ, characteristics starting further ahead travel slower than characteristics starting further behind. These characteristics converge and eventually cross, where the point solution ρ would become multi-valued, which is impossible. This means that a discontinuity, or shock, must form at the moment and location where characteristics first cross.

The Rankine-Hugoniot Condition

Once a shock has formed, separating a region of density ρ_B (behind) from a region of density ρ_F (forward), the conservation law must still hold in an integrated sense across the discontinuity. Integrating equation (2) across a thin strip and letting the strip's width go to zero yields the Rankine-Hugoniot condition for the shock's speed *s* (Garavello and Piccoli, 2006).

(10) $$s = \frac{q(\rho_B) - q(\rho_F)}{\rho_B - \rho_F}$$

This formula tracks a jam's boundary once it has formed and is going to be used later to derive the growth of the jam behind a red light and its clearance once the light turns green.

Rarefaction Waves and the Entropy Condition

Not every discontinuity that satisfies (10) is physically admissible. A shock is admissible only if characteristics run into it from both sides (the Lax entropy condition),

(11) $$q'(\rho_B) > s > q'(\rho_F)$$

Because *q'* is decreasing in ρ for the Greenshields flux, (11) holds only when $\rho_B < \rho_F$, that is, when density increases in the direction of travel, as at the back of a queue.

If instead density decreases in the direction of travel ($\rho_B > \rho_F$), characteristics diverge and a shock would violate (11). The solution is then a continuous **rarefaction fan**. For the Greenshields flux, setting $q'(\rho) = \frac{x}{t}$ for a fan centered at x = 0 gives

(12) $$\rho(x, t) = \frac{\rho_{max}}{2}\left(1 - \frac{x}{v_{max}t}\right)$$

between the characteristic speeds $q'(\rho_B)$ and $q'(\rho_F)$. The release of a standing queue is of this type: the jam density ρ_{max} lies upstream of a lower density, and the leading edge of the fan travels upstream at $q'(\rho_{max}) = -v_{max}$.

The Merge-Light Model

Consider a single-lane main road along the coordinate *x*, with a traffic signal and a merging on-ramp coinciding at the single point *x* = 0. When (*x* < 0), traffic arrives at a constant density ρ_{in}, corresponding inflow rate $q_{in} = q(\rho_{in})$ under the flux function of equation (5). At the same location, a ramp contributes additional vehicles to the road at a constant rate r_0 (vehicles per unit time). The constant inflow assumption is a simplification as a real ramp inflow fluctuates over time. However, this is standard practice for the first analytical calculations (Kolbe, 2024), and time-varying inflow is left to future work (Section 7). The signal cycles with period *T*, alternating between a red phase of duration τ_r and a green phase of duration $\tau_g = T - \tau_r$.

The ramp joins the main road exactly at the stop line x = 0, and ramp vehicles face the same signal as the main road. The configuration is therefore effectively a two-approach signalised junction feeding a single downstream lane, with no further bottleneck downstream.

The behavior of the model is affected entirely by what happens at $x = 0$, where the road's outflow $q(0^-, t)$ must account for both the red-green state of the signal and the competing demand from the main road and the ramp.

During the red phase ($0 \leq t < \tau_r$ within cycles), the signal blocks all outflow

$$q(0^-, t) = 0 \tag{13}$$

With no outflow permitted, density accumulates immediately upstream of $x = 0$ from both the main-road inflow q_{in} and the ramp inflow r_0.

During the green phase ($\tau_r \leq t < T$), the signal permits outflow up to the road's maximum flow capacity of $q_{max} = \frac{v_{max}\rho_{max}}{4}$ (from section 2). The actual outflow produced by the standard demand-supply formalism used at network junctions in the cell-transmission framework (Daganzo, 1995; Garavello and Piccoli, 2006):

$$q(0^-, t) = \min(\text{demand}(t), q_{max}) \tag{14}$$

where demand(t) is the combined rate at which the main road and ramp are attempting to send vehicles through $x = 0$.

Merge Rule: When both the main road and the ramp compete for the same green-phase capacity q_{max}, a rule is needed to divide that capacity between them. This paper uses that standard priority (right-of-way) merge of demand-supply junction models (Daganzo, 1995; Lebacque, 1996; Garavello and Piccoli, 2006): the two approaches are assigned fixed shares of q_{max}, and any capacity one approach cannot use is passed to the other. The shares are chosen in proportion to the nominal flows of the streams, as in the influx-ratio-preserving coupling of Kolbe (2024),

$$\alpha = \frac{q_{in}}{q_{in}+r_0} \tag{15}$$

This gives both approaches the same flow ratio $\frac{q_{in}+r_0}{q_{max}}$, an equal-saturation principle in signal timing. The rule is applied through the actual sending flows. Let $S_m(t)$ be the main-road sending flow, which is q_{max} (the sending capacity of the jammed queue) while a queue is present and q_{in} otherwise, and let $S_r(t)$ be

the ramp sending flow, which is its discharge capacity c_r while the ramp queue is non-empty and r_0 otherwise. The ramp lane is assumed to have $c_r \geq (1 - \alpha)q_{max}$, so its own capacity is never binding. The outflows are

(16) $q_{road} = min(S_m, max(\alpha q_{max}, q_{max} - S_r)), \quad q_{ramp} = min(S_r, max(1 - \alpha)q_{max}, q_{max} - S_m))$

Two regimes follow. If the combined free-flow demand is below capacity, $q_{in} + r_0 < q_{max}$, then (14) passes both streams unrationed and no division is needed. While both queues exist, $S_m = q_{max}$ and $S_r = c_r$, and (16) gives

(17) $$q_{road} = q_{max}\frac{q_{in}}{q_{in}+r_0}, \quad q_{ramp} = q_{max}\frac{r_0}{q_{in}+r_0}$$

The regime in which only one of the two queues remains does not arise before clearance, because section 4 shows that, under the shares (17), the main-road queue and the ramp queue empty at the same instant.

For clarity and to keep these calculations as well as derivations in section 4 manageable, the model adopts the following assumptions (each effect can be assessed later):

Single lane, no lane changing beyond the merge point so traffic downstream of $x = 0$ is treated as a single homogeneous stream

Constant free-flow speed v_{max} and jam density ρ_{max} and these parameters do not vary with location or time

Constant ramp inflow r_0 so the ramp contributes vehicles at a fixed rate rather than a time-varying one. This assumption is going to be most worth studying since real ramp demand fluctuates

Constant main-road inflow ρ_{in} and is steady rather than time-varying

Two kinds of queue

The main road is modelled as a spatial LWR queue, because its length is the quantity of interest and the queue spills back along the road. The ramp is modelled as a point (vertical) queue with unlimited storage, because only the number of waiting vehicles and their clearance time matter for the questions asked here, and a short ramp has little storage to spill back into. The two representations are therefore chosen for the quantity each is used to answer, and ramp spillback onto the main road is not captured.

Shocks and Shock Formation

During the red light phase, $q(0^{-}, t) = 0$ (equation 13), so no vehicles leave the main road at $x = 0$. The main-road cars and the ramp cars are treated separately, consistent with the assumption that ramp vehicles wait on the ramp itself rather than merging onto the main road until the traffic light allows.

Main Road

Immediately upstream of $x = 0$, density rises to the jam density ρ_{max} (the only density at which flux is zero other than $\rho = 0$, and the physically relevant one for stopped traffic). This creates a shock separating the approaching free-flow state ρ_{in} (behind, smaller x) from the jammed state ρ_{max} (ahead, at $x = 0$). The shock satisfies the entropy condition (11) since $\rho_{in} < \rho_{max}$. By the Rankine-Hugoniot condition (equation 10), the shock's speed is

(18) $$s_{main} = \frac{q(\rho_{in}) - q(\rho_{max})}{\rho_{in} - \rho_{max}} = \frac{q_{in}}{\rho_{in} - \rho_{max}}$$

using $q(\rho_{max}) = 0$. Since $\rho_{in} < \rho_{max}$, the denominator is negative, so $s_{main} < 0$. The shock moves in the negative x direction (back of the line retreats from the light).

Letting

(19) $$a = |s_{main}| = \frac{q_{in}}{(\rho_{max} - \rho_{in})}$$

for the rate at which the line lengthens, the line's position at the moment the red light ends ($t = \tau_r$) is $x_{back}(\tau_r) = -a\tau_r$, so the length of the stopped region at the end of red it

(20) $$L_{max} = a\tau_r = \frac{q_{in}}{\rho_{max} - \rho_{in}}\tau_r$$

L_{stop} is the length of the stopped region at the end of red. It is not the furthest upstream reach of the congestion, because the back of the line keeps retreating after the light turns green until the release wave arrives. That furthest reach, L_{max}, is derived in section 4.

Ramp

Since the vehicles on the ramp are assumed to wait on the ramp at a constant arrival rate r_0 with no release during the red light, the number of waiting ramp vehicles grows linearly

(21) $$N_{ramp}(t) = r_0 t, \qquad 0 \le t \le \tau_r$$

and reaching a maximum of $N_{ramp}(\tau_r) = r_0 \tau_r$ vehicles waiting at the moment the light turns green.

Green Light

At $t = \tau_r$, the light allows cars to pass, and the waiting vehicles begin leaving at the rate q_{road} (equation 17). Since $q_{road} < q_{max}$, the stop line caps the flux, and the discharge densities are the two roots of $q(\rho) = q_{road}$,

(22) $$\rho_2^{f,c} = \frac{\rho_{max}}{2}\left(1 \pm \sqrt{1 - \frac{q_{road}}{q_{max}}}\right)$$

with ρ_2^{f} the free-flowing density immediately downstream of the stop line and ρ_2^{c} the congested density immediately upstream of it (section 2). The jammed state ρ_{max} now lies upstream of the lower density ρ_2^{c}, so by section 2, the release is a rarefaction fan, not a shock: a shock here would violate the entropy condition (11). The leading edge of the fan travels upstream at $q'(\rho_{max}) = -v_{max}$, and its trailing edge at $q'(\rho_2^{c}) = -w$, where $w = v_{max}\left(\frac{2\rho_2^{c}}{\rho_{max}-1}\right) > 0$. Between the trailing edge and the stop line the density is ρ_2^{c}, and at x = 0 it jumps to ρ_2^{f}.

Physically, the back of the line does not technically "know" that the light has turned green until the starting wave reaches it. Since its motion is controlled by the Rankine-Hugoniot balance between ρ_{in} and ρ_{max} (equation 18), it continues retreating at the rate *a* even after $t = \tau_r$, until the starting wave catches up to it. When these two waves meet, the line is cleared.

Tracking both from the start of the red light phase (*t = 0*)

(23) $$x_{back}(t) = -at, \qquad x_{fan}(t) = -v_{max}(t - \tau_r) \text{ for } t \ge \tau_r$$

Setting $x_{back}(t^*) = x_{fan}(t^*)$ and solving for the meeting time *t**,

(24) $$-at^* = -v_{max}(t^* - \tau_r) \rightarrow at^* = v_{max}t^* - v_{max}\tau_r \rightarrow (v_{max} - a)t^* = v_{max}\tau_r$$

gives,

(25) $$t^* = \frac{v_{max}\tau_r}{v_{max} - a}$$

Since $a = \frac{v_{max}\rho_{in}}{\rho_{max}} < v_{max}$, the fan always reaches the back of the line. Measuring from the start of green ($t = \tau_r$), this takes

(26) $$s_1 = t^* - \tau_r = \frac{a\tau_r}{v_{max} - a}$$

At position $x^* = v_{max}s_1$. When the two waves meet, the line is not yet cleared. The back of the queue now interacts with the fan.

Interaction with the fan

After t* the back of the queue is a shock between ρ_{in} and the local fan density $\rho_R = \frac{\rho_{max}}{2}(1 - \frac{X}{v_{max}s})$, where X(s) is its position and $s = t - \tau_r$. By (10) its speed is

(27) $$\frac{dX}{ds} = v_{max}(1 - \frac{\rho_{in}+\rho_R}{\rho_{max}}) = \frac{k}{2} + \frac{X}{2s},\ k = q'(\rho_{in}) = v_{max}(1 - \frac{2\rho_{in}}{\rho_{max}})$$

This is a linear ODE. It has the particular solution X = ks and the homogeneous solution $\sqrt{s}$, and with $X(s_1) = -v_{max}s_1$ the solution is

(28) $$X(s) = ks + C\sqrt{s}, \quad C = -(v_{max} + k)\sqrt{s_1}$$

The back of the queue stops retreating when $\frac{dX}{ds} = k + \frac{C}{2\sqrt{s}} = 0$, that is at $\sqrt{s_m} = -\frac{C}{2k}$. The furthest upstream reach of the queue is therefore

(29) $$L_{max} = -X(s_m) = \frac{C^2}{4k} = \frac{(v_{max}+k)^2 s_1}{4k}, \qquad s_m = \frac{(v_{max}+k)^2 s_1}{4k^2}$$

This holds provided the zero-speed point lies inside the fan, which requires $\rho_{max} - \rho_{in} \geq \rho_2^c$, equivalent to $q_{road} \geq q_{in}$. Note that L_{max} does not depend on q_{road} and therefore not on the ramp inflow r_0, because the leading edge of the fan always travels at $-v_{max}$.

Clearance

After leaving the fan, the back shock moves forward and the queue is cleared when it reaches $x = 0$. The clearance time follows from conservation. Let $E(t) = \int\left(\rho - \rho_{in}\right) dx$ over the road upstream of the stop line be the excess vehicle count. By (2), $dE/dt = q_{in} - q\left(0^{-}, t\right)$, so after the start of green, while the queue exists and the flux is q_{road},

(30) $$E(t) = q_{in}\, t - q_{road}\left(t - \tau_r\right)$$

The queue has cleared when $E = 0$. Measured from the start of green,

(31) $$t_{clear} = \frac{q_{in}\tau_r}{q_{road} - q_{in}} = \tau_r \frac{X_s}{1 - X_s}, \qquad X_s = \frac{q_{in} + r_0}{q_{max}}$$

where the second form uses (17). Following the shock through the fan and then forward at constant speed $u = v_{max}\left(1 - \left(\rho_{in} + \rho_2^c\right)/\rho_{max}\right)$ gives the same value, which provides an independent check of (29) and (31).

The ramp line is treated more simply rather than the full shockwave analysis above. The discharge rate is q_{ramp} and arrival rate r_0, the ramp leaves at net rate $q_{ramp} - r_0$, so the clearance time is

(32) $$t_{clear}(ramp) = \frac{r_0 \tau_r}{q_{ramp} - r_0} = \tau_r \frac{X_s}{1 - X_s}$$

which similarly requires $q_{ramp} > r_0$ to clear it all, and which equals (31): the main-road queue and the ramp queue empty at the same instant.

The maximum main road line length is given by equation (29), and the main road clearance time from the start of the green light is given by equation (31). These are both expressed entirely in terms of q_{in}, r_0, ρ_{in}, τ_r, q_{road}, ρ_2^c and v_{max}. The ramp is represented using equations (21) and (32).

The full system has cleared only once both lines have disappeared, so the total clearance time is

(33) $$t_{clear}(total) = max\left(t_{clear},\, t_{clear}(ramp)\right)$$

Analysis and Special Cases

Non-clearing Condition

Equation (31) is finite when $q_{road} > q_{in}$ so the traffic jam will clear eventually given unlimited green light time. However, the green light time is not unlimited and only lasts τ_g before the light returns to red. This means

(34) $$t_{clear} \leq \tau_g$$

Substituting equation (31) and simplifying:

(35) $$\frac{q_{in}\tau_r}{q_{road}-q_{in}} \leq \tau_g \;\Leftrightarrow\; q_{in}\tau_r \leq \left(q_{road} - q_{in}\right)\tau_g \;\Leftrightarrow\; q_{in}\left(\tau_r + \tau_g\right) \leq q_{road}\tau_g$$

and this rearranges to

(36) $$q_{in}T \leq q_{road}\tau_g$$

This is a statement of vehicle conservation over one cycle: the vehicles arriving on the main road in a cycle, $q_{in}T$, cannot exceed the number the stop line can discharge during green, $q_{road}\tau_g$. If equation (36) fails, the queue does not fully clear before the red light appears, and the excess vehicle count at the end of each cycle grows by $q_{in}T - q_{road}\tau_g$ per cycle whatever the wave speeds. If it holds, every cycle starts with an empty queue and the solution is periodic. Using (17), $q_{in}/q_{road} = \left(q_{in} + r_0\right)/q_{max}$, so the condition becomes

(37) $$\frac{q_{in}+r_0}{q_{max}} \leq \frac{\tau_g}{T}$$

The same argument applied to the ramp gives the condition

(38) $$\frac{r_0}{q_{ramp}} \leq \frac{\tau_g}{T}$$

which is the same condition as (37), since $r_0/q_{ramp} = \left(q_{in} + r_0\right)/q_{max}$. The full system works when both equations (36) and (38) hold. For $r_0 = 0$, equation (37) reduces to $q_{in}/q_{max} \leq \tau_g/T$, the standard degree-of-saturation condition for a signalised link.

Other Cases

If there is no ramp ($r_0 \to 0$), equation (15) gives $\alpha \to 1$ and equation (17) gives $q_{road} \to q_{max}$ and the main road is the only one. Correspondingly, ρ_2 (solving $q\left(\rho_2\right) = q_{max}$) reduces to $\rho_2^f = \rho_2^c = \rho_{max}/2$, the density at which flux is maximized under the Greenshields relation from section 2. The trailing edge of

the fan then has speed $q'\left(\rho_{max}/2\right) = 0$, so it is stationary at the stop line and the discharge flux is q_{max}. This is the classical release structure for a signal-only link: a rarefaction fan with stop-line flux q_{max}. Equations (29) and (31) apply with $X_s = q_{in}/q_{max}$.

If there is no signal ($\tau_r \to 0$), there is no red phase so equation (20) gives $L_{stop} \to 0$ and equation (29) gives $L_{max} \to 0$. This means no line forms since the light never blocks outflow and there are no shocks formed. The merge-only case with $\tau_r \to 0$ and $q_{in} + r_0 > q_{max}$ is outside the present model and is not examined here.

Sensitivity Analysis on Each Variable

Holding a fixed, equations (20), (29) and (31) are all linear in τ_r, so doubling the red duration doubles the stopped-region length, the maximum line length and the clearance time.

The dependence on ramp inflow r_0 enters only through the share α: a larger r_0 decreases the main road's leaving rate q_{road}, which lowers the free-flow discharge density ρ_2^f and raises the congested discharge density ρ_2^c. The furthest reach L_{max} is unaffected, as equation (29) does not contain q_{road}. The clearance time does depend on r_0: X_s in equation (31) increases with r_0, so t_{clear} increases, and it grows without bound as $X_s \to 1$. A larger ramp inflow therefore always slows main-road clearance and tightens the stability margin in equation (37).

For a fixed signal timing, a larger ramp inflow can push a stable intersection into an overcrowded space. This is a practical implication because two intersections with identical signal timing and identical main-road traffic can have entirely different behavior depending on the amount of cars coming in on the ramp.

Equation (37) shows that for fixed T, decreasing τ_r (equivalently, increasing τ_g) always widens the stability margin and quantifies exactly how much green light time is needed to accommodate a given combination of ρ_{in} and r_0.

Numerical Verification

To test the analytical results of sections 4 and 5, the conservation law was also solved directly using the cell-transmission model (CTM), a Godunov-type finite volume scheme standard in many traffic flow literature and textbooks (Daganzo, 1994). Because the CTM is the Godunov scheme for the LWR conservation law, a converged CTM solution is the reference LWR solution, so agreement or disagreement with the analytic formulas tests the analysis and not the model.

Numerical Method

The road was separated into cells of width $\Delta x = 1$ m with time step $\Delta t = 0.05$ s (satisfying the condition $\Delta t \leq \Delta x / v_{max}$; the CFL number is $v_{max}\Delta t/\Delta x = 0.75$), using the demand-supply flux

$$F_{i+1/2} = min\left(S\left(\rho_i\right), R\left(\rho_{i+1}\right)\right) \tag{39}$$

at every meeting point. The details needed to reproduce the runs are as follows.

Domain: $x \in [-1500, 200]$ m, with the stop line at $x = 0$.

Boundary conditions: upstream, flux q_{in} enters the first cell; downstream, free outflow with flux $S\left(\rho_{last}\right)$.

Initial condition: $\rho = \rho_{in}$ everywhere, with the signal starting red at $t = 0$ (empty queue at the start of the first cycle).

Signal: at the stop-line interface the flux is zero during red, and during green it is given by the junction rule (16) together with the receiving function of the downstream cell.

Ramp: the ramp queue is a point queue updated by $N^{n+1} = N^n + \left(r_0 - q_{ramp}^n\right)\Delta t$ with q_{ramp}^n from (16) and $c_r = q_{max}$; it is coupled to the road only through the junction rule.

Cycles: multi-cycle runs use 10 cycles with no warm-up, since the initial state is the analytic free-flow state; first-cycle results correspond directly to the analysis in Section 4.

Extraction from the density field: L_{max} is the largest distance, over a cycle, from the stop line to the centre of the most upstream cell with $\rho > \rho_{max}/2$. L_{stop} is the same quantity at $t = \tau_r$. t_{clear} is the first time after the start of green at which the excess vehicle count $E(t) = \sum_i \left(\rho_i - \rho_{in}\right)\Delta x$ falls below 0.05 vehicles. The residual queue is E at the end of each cycle.

Convergence: results at $\Delta x = 2$, 1 and 0.5 m (CFL fixed) are reported in Table 4.

The code is available as supplementary material.

Test Parameters

Two parameter sets were used (Table 1). Set A is the original set. Set B has a more realistic ramp demand, which produces a ramp queue of several vehicles, and is undersaturated.

Table 1. *Test parameters.*

Quantity	Set A	Set B
v_{max}	15 m/s (54 km/h)	15 m/s (54 km/h)
ρ_{max}	0.15 veh/m (150 veh/km)	0.15 veh/m
ρ_{in}	0.03 veh/m	0.02 veh/m
r_0	0.04 veh/s (144 veh/h)	0.10 veh/s (360 veh/h)
τ_r, τ_g (T)	20 s, 40 s (60 s)	30 s, 60 s (90 s)
q_{max}	0.5625 veh/s (2025 veh/h)	0.5625 veh/s
q_{in}	0.36 veh/s (1296 veh/h)	0.26 veh/s (936 veh/h)
q_{road}, q_{ramp}	0.50625, 0.05625 veh/s	0.40625, 0.15625 veh/s
ρ_2^f, ρ_2^c	0.0513, 0.0987 veh/m	0.0355, 0.1145 veh/m
a	3.00 m/s	2.00 m/s
$\left(q_{in} + r_0\right)/q_{max}$ versus τ_g/T	0.711 > 0.667, equation (37) fails	0.640 < 0.667, equation (37) holds

Set A does not satisfy the stability condition (37): $q_{in}T = 21.6$ vehicles arrive per cycle but at most $q_{road}\tau_g = 20.25$ can leave, and the ramp condition (38) fails in the same way ($r_0/q_{ramp} = 0.711 > 0.667$, and equation (32) gives a ramp clearance time of 49.2 s, longer than $\tau_g = 40$ s). Set A is therefore oversaturated, and the main-road queue and ramp queue are both expected not to clear. Set B satisfies the condition and is expected to clear every cycle.

RESULTS

Table 2 compares the analytic and CTM values. For Set A the clearance time is taken from a first-cycle run with an extended green ($\tau_g = 200$ s), since the queue does not clear within the actual 40 s green. The difference is $(CTM - analytic)/analytic$.

Table 2. *Analytic predictions versus numerical (CTM) simulation results.*

Quantity	Analytic	Numerical (CTM)	Difference
Set A			
L_{stop}, eq. (20)	60.00 m	59.5 m	−0.8%
L_{max}, eq. (29)	80.00 m	78.5 m	−1.9%
t_{clear}, eq. (31) (extended green)	49.23 s	48.9 s	−0.7%
$N_{ramp,max}$, eq. (21)	0.800 veh	0.800 veh	0.0%
$t_{clear}(ramp)$, eq. (32)	49.23 s	did not clear within τ_g	consistent ($> \tau_g$)
Residual main-road queue, end of cycle 1	1.35 veh	1.357 veh	+0.5%
Residual ramp queue, end of cycle 1	0.15 veh	0.151 veh	+0.7%
Set B			
L_{stop}, eq. (20)	60.00 m	59.5 m	−0.8%
L_{max}, eq. (29)	70.91 m	70.5 m	−0.6%
t_{clear}, eq. (31)	53.33 s	53.0 s	−0.6%
$N_{ramp,max}$, eq. (21)	3.00 veh	3.00 veh	0.0%
$t_{clear}(ramp)$, eq. (32)	53.33 s	53.35 s	0.0%
Residual queue, end of every cycle	0	0 (all 10 cycles)	

Because Set A is oversaturated, the residual queue grows each cycle by $q_{in}T - q_{road}\tau_g = 1.35$ vehicles (main road) and $r_0T - q_{ramp}\tau_g = 0.15$ vehicles (ramp). Table 3 compares this with the CTM over several cycles.

Table 3. *Set A, main-road residual excess vehicles at the end of each cycle.*

Cycle	1	2	3	4	5	10
Analytic residual (veh)	1.35	2.70	4.05	5.40	6.75	13.50
CTM residual (veh)	1.357	2.707	4.057	5.407	6.757	13.507

Figure 1 shows the analytic space-time structure for Set B over one cycle, Figure 2 shows the CTM density field over several cycles for both sets, and Figure 3 overlays the analytic queue extent on the CTM result.

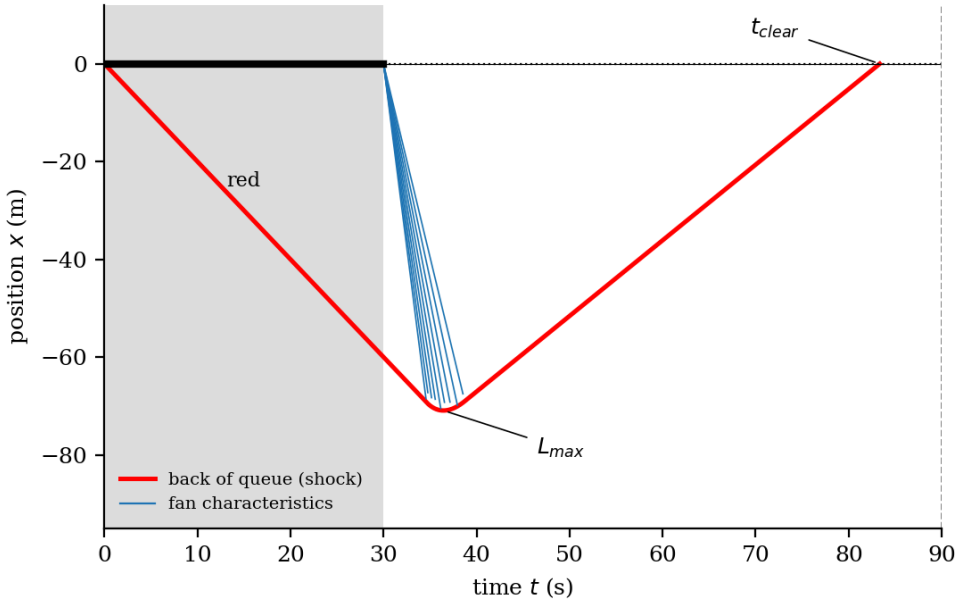


Figure 1. *Analytic wave structure for Set B over one cycle. The grey band is the red phase, the thick red line is the back of the queue, and the blue lines are characteristics of the release fan.*

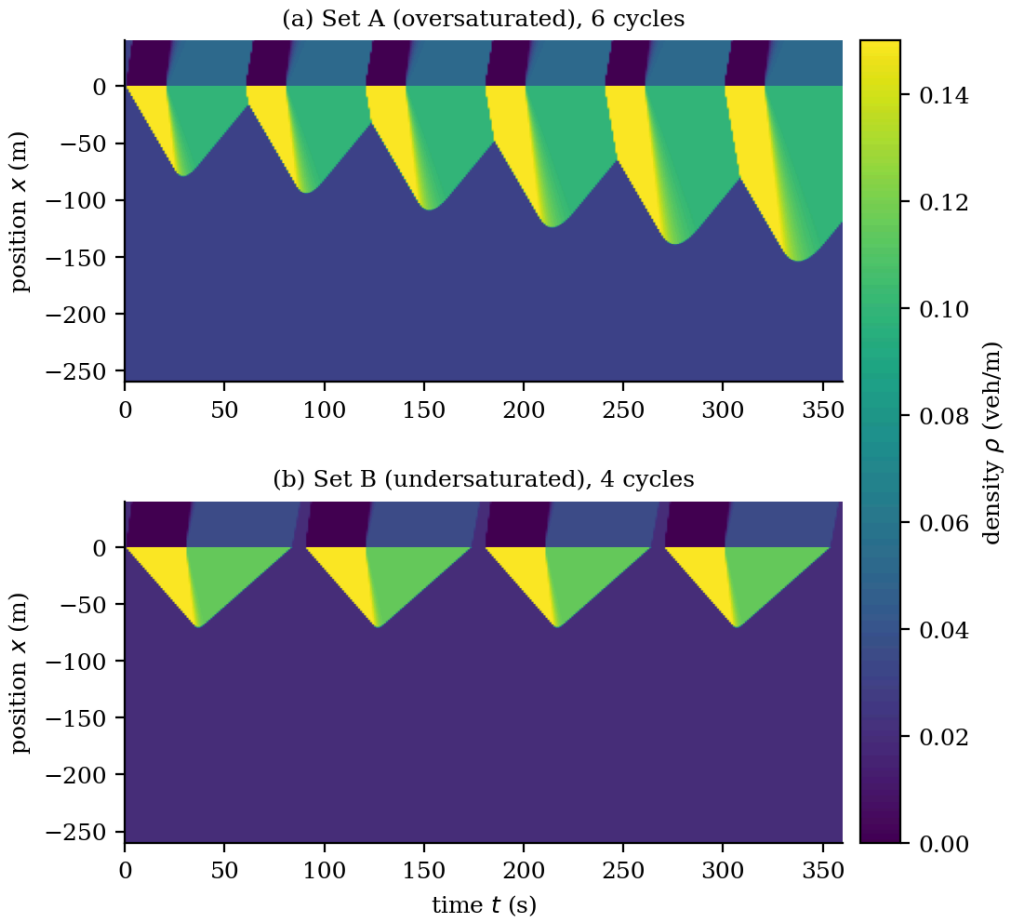


Figure 2. *CTM density fields. (a) Set A, oversaturated, six cycles. (b) Set B, undersaturated, four cycles.*

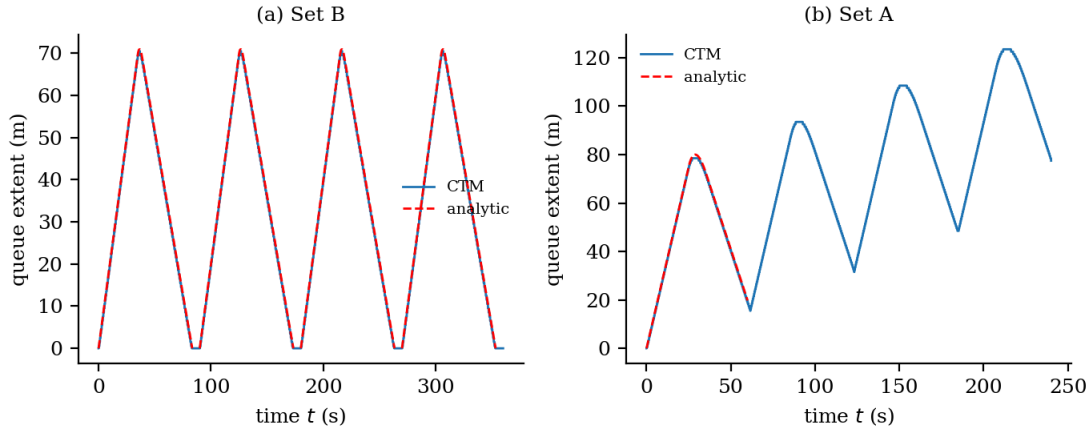


Figure 3. *Queue extent against time, CTM (solid) and analytic (dashed). (a) Set B, four cycles. (b) Set A: the analytic solution is plotted for the first cycle only, after which the oversaturated queue grows.*

Grid convergence. The extracted L_{max} changes by at most 1.5 m, about one cell, and t_{clear} is unchanged (Table 4).

Table 4. *Grid refinement (CFL number fixed at 0.75).*

Set	Δx (m)	L_{max} (m)	t_{clear} (s)
A (extended green)	2	79.0	48.9
	1	78.5	48.9
	0.5	79.25	48.9
B	2	69.0	53.0
	1	70.5	53.0
	0.5	70.25	53.0

Parameter sweeps. Starting from Set B, each of r_0, ρ_{in} and τ_r was varied in turn (first cycle, $\tau_g = 200$ s so that clearance is observed; points violating $q_{road} \geq q_{in}$ are omitted). Figure 4 shows the errors. The analytic L_{max} exceeds the CTM value by 0.4% to 5.1% over all three sweeps, an absolute difference of at most 1.5 m, consistent with the resolution of the threshold extraction. The estimate $a\tau_r$ (the stopped-region length) is below the CTM furthest reach by 5% to 36%. The analytic t_{clear} exceeds the CTM value by 0.3% to 1.9% wherever the CTM queue clears within the 200 s window; two points of the ρ_{in} sweep have analytic clearance times (331 s and 720 s) beyond the window, and correspondingly the CTM does not clear in 200 s. The sweep over r_0 also confirms that L_{max} does not depend on r_0 (analytic 70.9 m throughout; CTM 69.5 to 70.5 m), while t_{clear} rises from 29.7 s to 134.6 s.

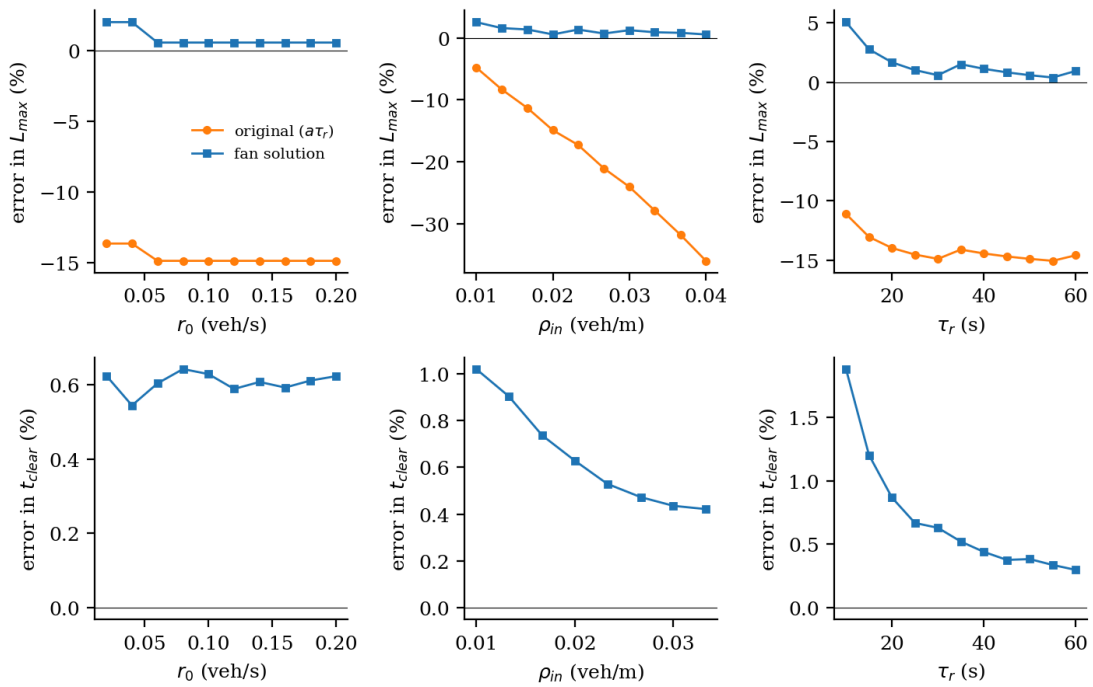


Figure 4. *Relative error of the analytic prediction against the CTM, for sweeps over* r_0, ρ_{in} *and* τ_r. *Top row: furthest reach* L_{max} *(equation (29), squares;* $a\tau_r$, *circles). Bottom row: clearance time.*

Stability map. Figure 5 compares the analytic stability boundary (37) with CTM outcomes on a grid of 21 values of r_0 and 19 values of τ_g/T, with $\rho_{in} = 0.02$ veh/m and $T = 90$ s. Each run used 8 cycles and $\Delta x = 2$ m. A run was classified unstable if the main-road excess or the ramp queue was at least 1 vehicle at the end of the eighth cycle. The analytic boundary agrees with the CTM classification in 397 of 399 cases. The two exceptions lie within 0.001 veh/s of the boundary, where the analytic growth per cycle is about 0.05 vehicles and has not reached the 1-vehicle threshold after 8 cycles.

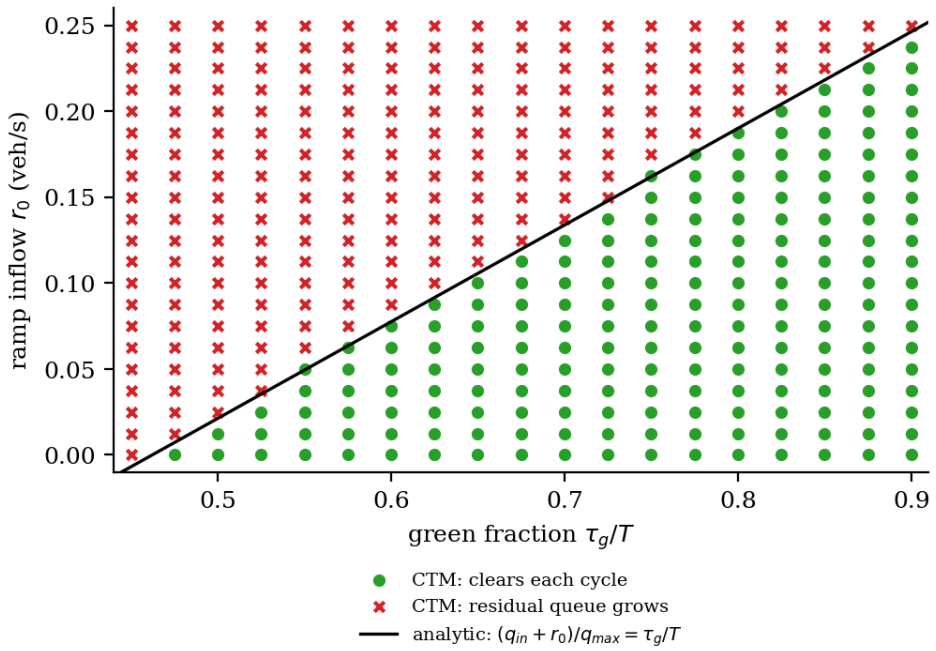


Figure 5. *Stability map in the* $\left(\tau_g/T,\ r_0\right)$ *plane. The line is the analytic boundary (37). Crosses: CTM residual queue grows; circles: CTM clears every cycle.*

Interpretation

The prediction from equation (21) matches the simulation essentially exactly. This is a consistency check of the implementation and not an independent validation, since equation (21) is a straightforward accumulation argument with no wave dynamics involved and the simulation applies the same accumulation rule. The main-road results are the substantive test, since they involve the wave dynamics, and they agree with the analysis to within about 2%.

In Set A the main-road queue does not clear within the green light phase. This is expected and is not a failure of the simulation: Set A is oversaturated, since more vehicles arrive per cycle (21.6) than can leave (20.25), so the queue must grow from cycle to cycle whatever the wave speeds. The jam length and clearance time in Set A are therefore compared in the first cycle and with an extended green, as in Table 2.

The CTM shows the discharge state settling on the branch immediately adjacent to the traffic light: cars are moving and leaving at the capped rate q_{road}, but at a density still above the critical density $\rho_{max}/2$ ($\rho_2^c \approx 0.0987$ veh/m in Set A), with the free-flow density $\rho_2^f \approx 0.0513$ veh/m on the downstream side. This is exactly the Riemann solution at a flux-constrained interface described in Section 2, and it follows directly from the conservation law. It is not evidence of capacity drop. Capacity drop is an empirical

phenomenon, in which the discharge flow observed downstream of a bottleneck is lower than the capacity measured before the queue forms (Hall and Agyemang-Duah, 1991; Cassidy and Bertini, 1999; Srivastava and Geroliminis, 2013), and a first-order model with a concave flux cannot reproduce it.

The wave structure, rather than any physical effect missing from the LWR model, accounts for the main-road results. Treating the release as a single shock with a free-flow discharge state does not satisfy the entropy condition (11). Treating it as a rarefaction fan whose leading edge travels at $-v_{max}$ gives a furthest reach of 80.0 m, which agrees with the CTM value of 78.5 m. This model captures the qualitative behavior (traffic jam formation, a stability threshold, the role of the ramp inflow) and also gives the jam length and clearance time to within about 2% of the simulation.

CONCLUSION AND FUTURE WORK

This paper derived closed-form expressions for the growth and clearance of a traffic jam at a location where a traffic light/signal and a merging on-ramp meet. This is very common in real road design but is comparatively underexplored in literature relative to either aspect studied alone. The central results are the maximum traffic jam length (equation (29)), the clearance time (equation (31)), and a stability condition (equation (37)) which shows that an intersection remains under-saturated only when the total vehicles arriving per cycle do not exceed the stop-line capacity during green, that is, when the flow ratio $\left(q_{in} + r_0\right)/q_{max}$ stays below the fraction of the cycle spent on the green light phase. The limiting-case checks in section 5 showed that, with no ramp, the model reduces to the classical signal-only release (a rarefaction fan with stop-line flux q_{max}), and that with no signal no queue forms; the merge-only case with $q_{in} + r_0 > q_{max}$ was not examined. Numerical verification in section 6 confirmed the model is accurate for both an oversaturated and an undersaturated parameter set, with queue extent and clearance time within about 2% of the simulation and a stability boundary that agreed with the simulation in 397 of 399 cases.

The next step is a work-conserving, time-dependent treatment of the junction in regimes where the two queues do not empty together, for example for arbitrary initial queues, and a justification of the chosen priority rule against data (Daganzo, 1995; Lebacque, 1996; Garavello and Piccoli, 2006).

This paper assumed a constant ramp arrival rate r_0 for tractability in section 3. Real ramp inflow r_0 fluctuates over the cycle of a traffic light and over the course of a day. Extending the model to a time-varying $r_0(t)$ would make equation (37) directly applicable to measured traffic.

The model also assumes a fixed cycle length and red light/green light split. Many real intersections instead use adaptive signals that adjust τ_r and τ_g in response to detected demand. Since equation (37) gives an explicit condition in terms of τ_r and τ_g, an extension is to treat the split as a control variable and explore which timing policies minimize average traffic jam length.

The LWR model assumes drivers adjust speed instantaneously based on local density alone with no reaction time or anticipation. Other models such as the Payne-Whitham framework (Payne, 1971; Whitham, 1974) relax this assumption and assess these factors, as does the Aw-Rascle-Zhang model (Aw and Rascle, 2000) as a modern second-order alternative.

This paper treated a single isolated traffic light and ramp merge. In real traffic, there are often multiple closely spaced ramps and the lanes can interact, where cars can move from one lane to another.

Comparison against real data. An accessible source for comparison against real data is the dataset of Genser et al. (2023), which provides 1-second resolution signal-state and loop detector data for an urban intersection in Zurich, Switzerland across 2 months. This particular intersection does not include a merging on-ramp, so it can be used to test the limiting case in section 5 where $r_0 \to 0$ rather than the full merging model. The intersection is regulated by a fully actuated signal, which conflicts with the fixed-cycle assumption of this paper, so a comparison would require extending the model to adaptive signals and is left to future work.

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