

A Formal Decomposition of the Partition Function at $n/2$

Mihir Kopalle

Abstract

This study splits the partition function $p(n)$ at the halfway point, $\lfloor x/2 \rfloor$, into two groups of partitions. Counting each group with finite sums, including restricted partitions, gives an exact formula for $p(n)$ that works for both odd and even values of n . The identity is checked against known values and offers a combinatorial way to understand partitions.

Introduction

Partition Function: A function called $p(n)$ that counts the number of distinct ways to write a non-negative integer as a sum of positive integers

Example: The number 4 can be written as 5 distinct partitions
Famous formulas for $p(n)$ have advanced structures

Problem Statement: Can a short, finite formula count them by splitting at the halfway point

Methods

For each partition of n , let λ be its largest part. We divide the partitions into two cases:

Group A: $\lambda \geq \lfloor x/2 \rfloor$:

The remainder $n - \lambda$ is no larger than λ . So it may be partitioned freely. Hence, the number of such partitions is $p(n - \lambda)$.

Group B: $\lambda < \lfloor x/2 \rfloor$:

The remainder $n - \lambda$ may contain parts larger than λ . These would violate the assumption that λ is the largest part and hence, we use a restricted partition function $p_{1,\lambda}(n - \lambda)$, counting only partitions of $n - \lambda$ whose parts are at most λ . Adding these counts over all possible values of λ gives $p(n)$.

$$\text{Group A} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p(i) \quad \Bigg| \quad \text{Group B} = \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n p_{1,n-i}(i)$$

Results

Therefore,

$$p(n) = \text{Group A} + \text{Group B}$$

$$p(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p(i) + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n p_{1,n-i}(i)$$

Discussions

- Unlike Euler's recurrence, this identity is based on the largest part of each partition, not generalized pentagonal numbers.

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

- Unlike Hardy-Ramanujan or Rademacher formulas, it is combinatorial and structural, rather than asymptotic or analytic.

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \frac{d}{dn} \left[\frac{1}{\sqrt{n - \frac{1}{24}}} \sinh \left(\frac{\pi}{k} \sqrt{\frac{2}{3} \left(n - \frac{1}{24} \right)} \right) \right]$$

- It separates $p(n)$ into ordinary and restricted partition counts, which makes the role of the largest part explicit.
- Its main value is interpretation of partition structure in the form of combinatorics, rather than faster computation.

Conclusion

- This work introduces a largest-part decomposition of $p(n)$ using ordinary and restricted partition functions
- The identity offers a distinct combinatorial viewpoint from classical recursive, asymptotic, and analytic formulas

Working Example

$$p(4) = 5$$

For Group A, $\lambda \geq 2$:

$$\begin{aligned} & p(4-4) + p(4-3) + p(4-2) \\ &= p(0) + p(1) + p(2) \\ &= 1 + 1 + 2 = 4 \end{aligned}$$

For Group A, $\lambda < 2$, so only $\lambda = 1$:

$$\begin{aligned} & p_{1,1}(3) + p_{1,0}(4) \\ &= 1 + 0 = 1 \end{aligned}$$

Partitions of $n = 4$

4
3 + 1
2 + 2
2 + 1 + 1
1 + 1 + 1 + 1

Therefore,
 $p(4) = 4 + 1 = 5$

$$p(4) = \sum_{i=0}^2 p(i) + \sum_{i=3}^4 p_{1,4-i}(i) = 5$$

References

- [1] G. E. Andrews, The Theory of Partitions. Cambridge: Cambridge University Press, 1998.
- [2] G. H. Hardy and S. Ramanujan, "Asymptotic formulae in combinatory analysis," Proceedings of the London Mathematical Society, vol. 17, pp. 75-115, 1918.
- [4] G. E. Andrews and K. Eriksson, Integer Partitions. Cambridge: Cambridge University Press, 2004.
- [3] H. Rademacher, "On the partition function $p(n)$," Proceedings of the London Mathematical Society, vol. 43, pp. 241-254, 1937.