

Quadratic Discriminant Solution for the Complex Roots of a Pure Cubic Equation

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ABSTRACT

This paper treats a single family of cubic equations lying at the intersection of two standard topics in elementary algebra: the quadratic discriminant $b^2 - 4ac$ and the roots of the cubic. For any real y and any nonzero real w , the difference-of-cubes factorization of $w^3x^3 - y^3$ yields a quadratic factor $w^2x^2 + (wy)x + y^2$ whose discriminant, with linear coefficient $b = wy$, equals $-3b^2$, equivalently $3b^2i^2$ with $i^2 = -1$. When $y \neq 0$, the two non-real roots of the cubic follow as scalar multiples of the primitive cube roots of unity $\omega = (-1 + i\sqrt{3})/2$ and $\omega^2 = (-1 - i\sqrt{3})/2$. The construction recovers, in this restricted setting, the classical fact that the third cyclotomic polynomial $\Phi_3(x) = x^2 + x + 1$ has discriminant -3 [1,10]. That fact is not new, and the contribution offered here is expository: an elementary, resolvent-free derivation that avoids the depression and resolvent steps of the general Cardano method, reduces all root extraction to a single quadratic discriminant, exhibits the imaginary unit explicitly at every stage, and makes the passage from the quadratic discriminant to the cube roots of unity visible to a reader who has met only the quadratic formula.

INTRODUCTION

The general cubic $ax^3 + bx^2 + cx + d = 0$ was first solved in radicals during the early sixteenth century. Scipione del Ferro is credited with a solution to one form of the depressed cubic, $x^3 + px = q$, around 1515 [2,3]; the same form was solved independently by Niccolò Fontana Tartaglia in his 1535 contest with del Ferro's student Antonio Fior [3]. Tartaglia communicated his method to Girolamo Cardano in 1539 under an oath of secrecy. After examining del Ferro's prior unpublished solution in Bologna in 1542, Cardano considered the oath voided and published a generalized treatment, with credit to del Ferro, Tartaglia, and his student Lodovico Ferrari, in *Ars Magna* (1545) [3,4]. The method requires a substitution to depress the cubic, a resolvent quadratic, and the extraction of two cube roots [4,5].

Cardano observed that the radical formula sometimes required the square root of a negative quantity, even for cubics with three real roots. The reconciliation of this paradox was supplied by Rafael Bombelli's *L'Algebra* (1572). For the equation $x^3 = 15x + 4$, Cardano's formula returns $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$. Bombelli observed that $(2 \pm i)^3 = 2 \pm 11i = 2 \pm \sqrt{-121}$, so the cube roots are $2 \pm i$, and the formula yields the real root $x = 4$ [6]. This calculation gave the first concrete demonstration that algebraic manipulation of conjugate complex intermediates produces a correct real solution, and it provides the foundation for the explicit use of i in the present derivation. The phenomenon of cubics with three real roots whose radical expression nevertheless requires complex intermediates was later termed *casus irreducibilis*, the irreducible case. Pierre Wantzel (1843) proved that for an irreducible cubic over the rationals with three distinct real roots, no expression of the roots in real radicals exists — complex intermediates are unavoidable [7,8].

The general cubic discriminant

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 \quad (1)$$

distinguishes three regimes for real coefficients: $\Delta > 0$ yields three distinct real roots (*casus irreducibilis* when the cubic is irreducible over the rational numbers); $\Delta = 0$ yields a repeated root; $\Delta < 0$ yields one real root and a conjugate pair of non-real complex roots [5,9]. A general cubic may be reduced by a linear change of variable $x \mapsto x - b/(3a)$ to a depressed cubic (one with no x^2 term), a technique due to Cardano [4].

While the general cubic admits Cardano's radical formula, manual application is cumbersome and, in *casus irreducibilis*, requires complex-number arithmetic to recover real roots [4,6,9]. Specific cubic forms therefore continue to admit individual analyses. The cubic studied here, $w^3 x^3 = y^3$ for real y and nonzero real w , is the simplest depressed cubic — a *pure cubic*, also called a *binomial cubic*, of the form $X^3 = c$ — and, when $y \neq 0$, falls in the regime $\Delta < 0$. It admits a closed-form factorization through the difference-of-cubes identity, after which the imaginary unit appears in the complex roots through the discriminant of the resulting quadratic factor.

Contribution and scope. The identity established below is not new. That the third cyclotomic polynomial has discriminant -3 is standard [1,10], and the roots of a pure cubic are obtained at once by taking a real cube root and multiplying by the three cube roots of unity. What this paper claims is narrower and expository: an elementary, resolvent-free derivation of that known special case for the family $w^3 x^3 = y^3$, together with a transparent link between the quadratic discriminant and the third cyclotomic polynomial. Its intended value is pedagogical. The argument uses nothing beyond a factorization identity and the quadratic formula, yet it produces the cube roots of unity, the sign of the discriminant, and the field-theoretic meaning of the number -3 in one short computation, and it can therefore serve as a clean classroom bridge from the quadratic discriminant, which students meet early, to the cube roots of unity, which they typically meet much later. No new theorem is claimed, and the derivation does not displace the Cardano method for the general cubic.

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METHODS

The transformation used here parallels Cardano's resolvent step. In Cardano's method, the depressed cubic $t^3 + pt + q = 0$ is solved by the substitution $t = u + v$, producing a resolvent quadratic in u^3 with discriminant $-4p^3 - 27q^2$, whose sign determines the nature of the roots [4,5,9]. In place of a substitution, the present approach applies the difference-of-cubes identity $j^3 - k^3 = (j - k)(j^2 + jk + k^2)$ directly to the pure cubic, with $j = wx$ and $k = y$. The resulting quadratic factor, after the substitution $u = wx/y$ and division by y^2 , equals

$$\Phi_3(u) = u^2 + u + 1, \quad (2)$$

the third cyclotomic polynomial, whose discriminant is -3 [1,10]. Modern algebraic treatments unify the procedure within the theory of Lagrange resolvents, which are linear combinations of the roots weighted by roots of unity, formed so that they transform predictably under permutation of the roots: for a cubic with roots r_1, r_2, r_3 the resolvent $r_1 + \omega r_2 + \omega^2 r_3$ is fixed by the cyclic subgroup of order three of the Galois group, and its cube is rational in the coefficients [5,8]. The pure cubic studied here is the case in which two of the three resolvents vanish identically, so that no resolvent extraction is required.

The polynomial $\Phi_3(x) = x^2 + x + 1$ is irreducible over the rational numbers. The standard proof applies Eisenstein's criterion — a sufficient test for irreducibility, in which some prime divides every non-leading coefficient, does not divide the leading coefficient, and whose square does not divide the constant term — at $p = 3$ to the substituted polynomial $\Phi_3(x + 1) = x^2 + 3x + 3$, whose constant and linear coefficients are divisible by 3 but not 9, and whose leading coefficient is not divisible by 3 [1,8]. Irreducibility implies that Φ_3 is the minimal polynomial of ω over the rational numbers, so the cyclotomic field obtained by adjoining ω to the rational numbers is a degree-two extension, whose field discriminant is -3 [10].

The quadratic discriminant is also expressible as a resultant. For a polynomial $f(x) = ax^2 + bx + c$ with derivative $f'(x) = 2ax + b$, the resultant satisfies $\text{Res}(f, f') = -a(b^2 - 4ac)$, so $\text{disc}(f) = -\text{Res}(f, f')/a$ [5,9]. Applied to the quadratic factor of Eq. (5), this provides an alternative route to Eq. (8) without recourse to the quadratic formula.

PROPOSITION

Let y be real and let w be a nonzero real number. The quadratic factor of $w^3x^3 - y^3 = 0$, regarded as a polynomial in x , has discriminant $\Delta = -3(wy)^2$, equivalently $\Delta = 3b^2i^2$ with $b = wy$.

PROOF

For any real y and any nonzero real w , let

$$w^3 x^3 = y^3. \quad (3)$$

Eq. (3) is rewritten in standard form as

$$(wx)^3 = y^3, \quad \text{equivalently} \quad w^3 x^3 - y^3 = 0. \quad (4)$$

Applying the difference-of-cubes identity $j^3 - k^3 = (j - k)(j^2 + jk + k^2)$ with $j = wx$ and $k = y$, Eq. (4) becomes

$$(wx - y)[w^2 x^2 + (wy)x + y^2] = 0. \quad (5)$$

The discriminant Δ of the quadratic factor $w^2 x^2 + (wy)x + y^2$ in Eq. (5) is

$$\Delta = (wy)^2 - 4(w^2)(y^2). \quad (6)$$

Eq. (6) is rearranged as

$$\Delta = (wy)^2 - 4(wy)^2, \quad (7)$$

which simplifies to

$$\Delta = -3(wy)^2. \quad (8)$$

In the standard form $ax^2 + bx + c$, the coefficient of x in the quadratic factor of Eq. (5) is $b = wy$, so Eq. (8) may be written

$$\Delta = -3b^2. \quad (9)$$

Using $i^2 = -1$, Eq. (9) is equivalent to

$$\Delta = 3b^2 i^2 = 3(wy)^2 i^2. \quad (10)$$

Hence for any real y and any nonzero real w , in $w^3 x^3 = y^3$, the quadratic discriminant of the associated quadratic factor is $\Delta = 3b^2 i^2$, with $b = wy$, which is the assertion of the Proposition.

Assume from here that $y \neq 0$ as well, so that $\Delta = -3(wy)^2 < 0$; one root of the cubic is then real and the other two are non-real complex conjugates [5,9]. (For $y = 0$ the discriminant vanishes and Eq. (3) has the triple root $x = 0$.) The real root is obtained from the linear factor $wx - y = 0$ in Eq. (5):

$$x_1 = \frac{y}{w}. \quad (11)$$

The two complex roots follow from $w^2x^2 + (wy)x + y^2 = 0$ by the quadratic formula, with Δ taken from Eq. (10):

$$x_{2,3} = \frac{-wy \pm \sqrt{\Delta}}{2w^2} = \frac{-wy \pm \sqrt{3(wy)^2 i^2}}{2w^2}. \quad (12)$$

Eq. (12) simplifies to

$$x_{2,3} = \frac{-wy \pm wy i\sqrt{3}}{2w^2} = \frac{y(-1 \pm i\sqrt{3})}{2w}. \quad (13)$$

Eq. (13) reduces further to

$$x_{2,3} = \frac{y(-1 \pm i\sqrt{3})}{2w}. \quad (14)$$

The two complex roots of the cubic are therefore

$$x_2 = \frac{y(-1+i\sqrt{3})}{2w}, \quad x_3 = \frac{y(-1-i\sqrt{3})}{2w}. \quad (15)$$

Equivalently, $x_2 = (y/w)\omega$ and $x_3 = (y/w)\omega^2$, where $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$ is a primitive cube root of unity [1,10].

DISCUSSION

The factorization in Eq. (5) and the discriminant computation in Eqs. (6)–(9) recover, by elementary means, a special case of the classical identity $\text{disc}(\Phi_3) = -3$, where $\Phi_3(x) = x^2 + x + 1$ is the third cyclotomic polynomial [1,10]. The substitution $u = wx/y$ converts the quadratic factor of Eq. (5) into $y^2\Phi_3(u)$. Because the discriminant of $cp(x)$ for a quadratic p scales as c^2 , and the change of variable contributes a further factor of w^2 , the result of Eq. (8), $\Delta = -3(wy)^2$, is the expected scaling of the underlying -3 . The complex roots of Eq. (15) are accordingly y/w times the primitive cube roots of unity, in agreement with the standard observation that for any cubic of the form $X^3 = \alpha$ with $\alpha \neq 0$ the three roots are $\alpha^{1/3}$, $\alpha^{1/3}\omega$, and $\alpha^{1/3}\omega^2$ [5,9].

Three independent checks confirm Eq. (8) and the root formulas of Eq. (15); a fourth route, through the resultant, is indicated in the Methods.

(i) Vieta's formulas. The roots of Eq. (15) satisfy Vieta's formulas for the cubic $x^3 - (y/w)^3 = 0$. Using $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$,

$$x_1 + x_2 + x_3 = \frac{y}{w}(1 + \omega + \omega^2) = 0, \quad (16)$$

$$x_1x_2 + x_1x_3 + x_2x_3 = \frac{y^2}{w^2}(\omega + \omega^2 + \omega^3) = 0, \quad (17)$$

$$x_1x_2x_3 = \frac{y^3}{w^3}\omega^3 = \frac{y^3}{w^3}, \quad (18)$$

which match the coefficients of $x^3 - (y/w)^3 = 0$ exactly [5,9].

(ii) The general cubic discriminant. Applying Eq. (1) directly to $w^3x^3 - y^3 = 0$, with $a = w^3$, $b = c = 0$, and $d = -y^3$, all cross terms vanish and

$$\Delta_{cubic} = -27a^2d^2 = -27w^6y^6 = -27(wy)^6, \quad (19)$$

which is negative for any nonzero real y and any nonzero real w , confirming the regime of one real root and a conjugate pair of non-real complex roots [5,9]. The two discriminants are consistent with one another through the multiplicative identity $disc(fg) = disc(f) \cdot disc(g) \cdot Res(f, g)^2$ [5]: reducing to monic factors over the field obtained by adjoining y/w to the rational numbers, the linear factor contributes trivially, the quadratic factor contributes $-3(y/w)^2$, and the resultant of the two factors equals $3(y/w)^2$, which together recover $-27(y/w)^6$, matching Eq. (19) after the standard scaling by powers of the leading coefficient w^3 .

(iii) A numerical instance. Setting $w = 1$ and $y = 2$ reduces Eq. (3) to $x^3 = 8$. The quadratic factor of Eq. (5) becomes $x^2 + 2x + 4$, with discriminant $\Delta = 4 - 16 = -12$, while Eq. (8) gives $-3(wy)^2 = -3(2)^2 = -12$. The roots are $x_1 = 2$, $x_2 = -1 + i\sqrt{3}$, and $x_3 = -1 - i\sqrt{3}$, which are exactly 2 , 2ω , and $2\omega^2$, in agreement with Eq. (15).

The splitting field of $w^3x^3 - y^3$, that is, the smallest field containing all three roots, may be described in the same terms. Over the base field obtained by adjoining the real root y/w to the rational numbers, the splitting field is the further extension obtained by adjoining ω , equivalently $\sqrt{-3}$. The minimal polynomial of ω over this base field is precisely $\Phi_3(x)$, so the splitting field is a degree-two extension with Galois group $Z/2Z$, generated by complex conjugation [5,8,10]. This is the field-theoretic content of the elementary computation given above.

Geometrically, the three roots of Eq. (15) lie on a circle of radius $|y/w|$ in the complex plane, separated by angles of $2\pi/3$. Writing $\omega = e^{2\pi i/3}$, the roots are $(y/w)e^{2\pi ki/3}$ for $k = 0, 1, 2$, an instance of the general fact that the n th roots of a nonzero complex number form the vertices of a regular n -gon centered at the origin [5,10]. The trigonometric reformulation of Eq. (15) by Euler's formula gives

$x_{2,3} = (y/w)[\cos(2\pi/3) \pm i\sin(2\pi/3)]$, which evaluates to $(y/w)(-1/2 \pm i\sqrt{3}/2)$, again in agreement with Eq. (15).

Because $\Delta < 0$ here, the present cubic lies outside *casus irreducibilis*, which requires three distinct real roots and $\Delta > 0$ [7,8,9]. For $w^3 x^3 = y^3$ with real coefficients and $y \neq 0$, two of the three roots are necessarily non-real, and the imaginary unit appears in the closed-form root expressions for structural reasons rather than as a removable intermediate of the kind first reconciled by Bombelli [6].

A comparison with Cardano's general procedure shows what is, and what is not, gained. Cardano's method requires a depression substitution, the formation of a resolvent quadratic, the extraction of two cube roots of (in general) complex numbers, and a recombination via $t = u + v$ subject to the constraint $3uv = -p$ [4,5,9]. The derivation given here replaces those steps, for the pure cubic only, with a single difference-of-cubes factorization and a single quadratic-discriminant evaluation, the cube roots of y^3 being encoded as multiplication by 1, ω , and ω^2 in Eq. (15), so that no cube root of a complex quantity is taken explicitly. The saving should not be overstated. The pure cubic $w^3 x^3 = y^3$ is already the cubic most easily solved by inspection: one extracts the real cube root y/w and multiplies it by 1, ω , and ω^2 to obtain all three roots immediately. The difference-of-cubes route and the direct extraction route are therefore essentially equivalent in effort and are in a precise sense the same computation written two ways. The value of the discriminant approach is conceptual transparency rather than computational saving: it exhibits where the imaginary unit enters, namely as the square root of the negative discriminant $-3(wy)^2$ of the quadratic cofactor, and it identifies that cofactor as a scaled copy of the third cyclotomic polynomial. A reader who knows the quadratic formula but not the cube roots of unity can thereby obtain the second from the first.

The approach taken here is specific to the pure cubic $w^3 x^3 = y^3$ and does not extend to general cubics with nonzero quadratic, linear, or constant terms not aligned with the difference-of-cubes factorization. For an arbitrary depressed cubic $t^3 + pt + q = 0$ with $q \neq 0$, the analogous resolvent quadratic $z^2 + qz - p^3/27 = 0$ still requires extraction of cube roots of its (possibly complex) solutions to recover the roots of the original cubic [5,9]. The present derivation therefore does not displace the Cardano method for the general case; it provides a discriminant-based account of the pure-cubic family, with the imaginary unit made explicit at every step and the connection to the third cyclotomic polynomial made transparent.

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