

Scenario-Dependent Closed-Loop Validity of the Clohessy-Wiltshire Model for Proportional-Derivative-Controlled Low Earth Orbit Rendezvous

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ABSTRACT

Space rendezvous and docking are critical for orbital debris removal, deep-space exploration, and in-orbit refueling. The Clohessy-Wiltshire (CW) model is widely employed for Low Earth Orbit (LEO) rendezvous due to its simplicity in estimating the relative motion of a chaser spacecraft. However, most existing studies assess the accuracy of the CW model in open-loop motion, leaving its performance under continuous feedback largely unexplored. While model discrepancies have the potential to accumulate during open-loop propagation, it remains unclear if these simplified models remain reliable during actively controlled maneuvers. To address this gap, this study propagates CW and nonlinear trajectories independently using an identical Proportional-Derivative (PD) controller and gain set. The analysis was conducted across various initial separations and offset directions, quantifying the maximum trajectory difference over a single orbit. By utilizing a two-body nonlinear reference, the linearization error was isolated, while the effects of J_2 perturbations and atmospheric drag were investigated separately.

Validation via NASA's General Mission Analysis Tool (GMAT) demonstrated a discrepancy of less than 1 μm in two-body relative motion. Results indicate that radial trajectories remained within 1 m for initial separations up to 5.5 km under feedback, compared to 95 m in free drift. The corresponding closed-loop thresholds were approximately 7.7 km along-track and 10 km cross-track, as opposed to the 600 m and 850 m in free drift. These thresholds are scenario-specific and shift with the selected error criterion, available thrust, controller bandwidth and radial energy matching, and must not be interpreted as universal CW validity bounds. The findings reveal that in the tested radial cases, available thrust became insufficient to counteract the induced along-track drift. Under the tested conditions, continuous feedback significantly extended the agreement between the linear and nonlinear models relative to free drift.

Keywords: spacecraft rendezvous; CW equations; proportional derivative (PD) law; closed loop guidance; propagator validation; Local Vertical Local Horizontal (LVLH), low Earth orbit (LEO)

1. INTRODUCTION

Spacecraft rendezvous refers to the controlled approach of one spacecraft, known as the chaser, towards another spacecraft, known as the target. Rendezvous requires the chaser to approach the target while reducing their relative position and velocity towards the required terminal conditions. Guidance and control studies hence depend on models which predict this relative motion as controller-generated acceleration commands are being applied (Woffinden and Geller). While nonlinear propagation represents the motion more completely, linear models such as Clohessy-Wiltshire (CW) retain their usefulness in providing preliminary analysis simply because they are easier to interpret and evaluate. The CW equations assume a circular orbit, point masses, negligible perturbations and small separation relative to orbital radius (Cliff 1-3; Clohessy and Wiltshire; Fehse; Hill).

Most studies of the CW model accuracy investigate open-loop motion, where no continuous feedback acts across the relative state. Earlier works have examined this accuracy within open-loop motion (Alfriend and Yan; Karlgaard and Lutze; Sullivan et al.; Willis et al.) or developed some extensions for eccentricity, higher-order effects and orbital perturbations (Carter and Humi; Gim and Alfriend; Schweighart and Sedwick; Yamanaka and Ankersen). Another case is where other studies have primarily developed rendezvous controllers using model-predictive (Di Cairano et al.; Weiss et al.), optimal (Moon et al.), impulsive (Hablanian et al.; Muldrow et al.) and low-thrust approaches (Kumar). However, open-loop model accuracy and closed-loop model agreement are not the same question. Under feedback, each controller repeatedly responds to the state predicted using its own model. Therefore, the differences between CW and nonlinear motion may be corrected before they gain a chance to accumulate, or may generate different control commands and lead to different outcomes. By itself, an open-loop accuracy boundary does not show how far the CW remains consistent with nonlinear motion during a controlled rendezvous.

Therefore, this study aims to test the hypothesis that closed-loop control could maintain the CW-to-nonlinear agreement over larger initial separations than free drift, while the resulting boundary was also expected to depend on its initial state, available thrust and offset direction. To address this gap, the CW and nonlinear trajectories were propagated independently while still being under the same continuous Proportional-Derivative (PD) law, acceleration limit and controller gains. In order to isolate the effect of choosing a model, without forcing the trajectories to share a command or state, each controller only used the position and velocity produced by its respective model. To test this hypothesis, this work varied the initial separation and Local Vertical Local Horizontal (LVLH) direction to measure the maximum difference in relation to the controlled trajectories. Using a two-body propagator in the main sweep, the CW linearisation error was isolated while J_2 , the second zonal harmonic associated with Earth's oblateness, and atmospheric drag were examined separately. To validate the nonlinear reference model, its results were compared against NASA's General Mission Analysis Tool (GMAT). The resulting comparison was then used to identify the tested conditions in which CW retained its suitability for preliminary closed-loop rendezvous analysis and also when nonlinear verification was necessary.

Table 1. Comparison of selected studies

Study	Control	Perturbations	External validation	Category
Sullivan et al.	none	2-body, J_2 (up to full-force)	none	Open-loop fidelity study
Schweighart and Sedwick	none	J_2	none	Perturbation-aware model
Hablani et al.	impulsive	none	none	Impulsive rendezvous scheme
Di Cairano et al.	continuous MPC	none	none	Control design, not validity study
Kumar	continuous low thrust	none (nonlinear 2-body)	none	Continuous low-thrust comparison
Muldrow et al.	impulsive	none (2-body)	none	Impulsive mismatch study
Present work	continuous PD (saturated)	J_2 , drag	NASA GMAT	Closed-loop direction sweep

2. METHODOLOGY

2.1. Local Vertical Local Horizontal (LVLH) Frame

Relative motion is naturally expressed in a target-centred LVLH frame whose basis vectors (1a) - (1c) have been constructed from the target's inertial position 'r' and its velocity 'v':

$$\hat{x} = \frac{r}{\|r\|}, \quad (1a)$$

$$\hat{z} = \frac{r \times v}{\|r \times v\|}, \quad (1b)$$

$$\hat{y} = \hat{z} \times \hat{x} \quad (1c)$$

In this triad, $\hat{x}$ is the radial axis (pointing away from Earth), $\hat{z}$ is the cross-track axis (the orbit normal), and $\hat{y}$ is the along-track axis, which completes the right-handed set.

The relative position of the chaser is $\rho = (x, y, z)$: positive x places the chaser farther from Earth than the target, positive y places it ahead along the orbit, and z is out of the orbital plane. Because the frame is tied to the moving target, it rotates with angular velocity $\omega = (r \times v)/\|r\|^2$ directed along $\hat{z}$. For a circular reference orbit of radius r its magnitude is the mean motion

$$n = \sqrt{\frac{\mu}{r^3}}, \quad (2)$$

where μ is Earth's gravitational parameter. For a circular reference orbit, ' r ' is constant, so both ' n ' and ' ω ' have to be constant as well. The euler acceleration term $\omega \times \rho$ therefore vanishes, effectively producing linear equations that have constant coefficients.

2.2. Nonlinear Reference Dynamics

Both vehicles have to follow two-body dynamics in an inertial frame in the two-body reference model, so the control acceleration will only be applied to the chaser. The target, which does not thrust, satisfies

$$\ddot{r}_t = \frac{-\mu r_t}{\|r_t\|^3}, \quad (3a)$$

and the chaser, which can apply a control acceleration a_c , satisfies

$$\ddot{r}_c = -\frac{\mu r_c}{\|r_c\|^3} + a_c. \quad (3b)$$

By writing:

$$\rho = r_c - r_t, \quad (3c)$$

and subtracting gives the exact relative acceleration in the inertial frame,

$$\ddot{\rho} = -\mu \left(\frac{r_t + \rho}{\|r_t + \rho\|^3} - \frac{r_t}{\|r_t\|^3} \right) + a_c. \quad (4)$$

No linearisation has yet been introduced. The term inside of the brackets here refers to the differential (or tidal), gravitational acceleration between the chaser and its target. Hence, the reference model propagates their inertial states separately under the selected force model, and then differences the states in order to transform the result into the LVLH frame. In terms of two body dynamics, the equivalent of this is essentially integrating this relative equation seen above; by adding J_2 , or drag it simply included their differential contributions inside of the nonlinear reference motion.

2.3. Linearisation and the Clohessy-Wiltshire (CW) Equations

Transforming the inertial relative acceleration into the rotating LVLH frame introduces the Coriolis term $2\omega \times \dot{\rho}$, the centrifugal term $\omega \times (\omega \times \rho)$, and an Euler term $\dot{\omega} \times \rho$ that vanishes because $\dot{\omega} = 0$ for a circular orbit. The remaining nonlinearity is the differential gravity, which the CW model replaces by its first-order behaviour: expanding $\|r_t + \rho\|^{-3}$ to first order in ρ leaves a neglected leading term of order $(\rho/r)^2$, that is, second order in the relative displacement (Willis et al. 8; Karlgaard and Lutze 23). Assembling the linearised differential gravity with the Coriolis and centrifugal terms and adding the control acceleration $a = (a_x, a_y, a_z)$ gives the CW equations (Cliff 2-3), with x radial, y along-track and z cross-track:

$$\ddot{x} = 3n^2 x + 2n\dot{y} + a_x, \quad (5a)$$

$$\ddot{y} = -2n\dot{x} + a_y, \quad (5b)$$

$$\ddot{z} = -n^2 z + a_z. \quad (5c)$$

These equations are set to form the linear system $\dot{s} = As + Ba$ in the state, where $s = (x, y, z, \dot{x}, \dot{y}, \dot{z})$ and, for the unforced case, the solution turns out as $s(t) = \Phi(t)s(0)$ (MIT OpenCourseWare). The Coriolis terms couple the radial and along track motion so that a chaser does not generally approach it in a straight line; the radial term $3n^2 x$ is a net outward stiffness, so a radial displacement tends to grow. Since the along-track equation has no term that restores its direct position, the bounded motion has to thereby depend on the initial state phasing instead.

2.4. Domain of Validity

Notice that the leading neglected term scales with $(\rho/r)^2$, so Clohessy-Wiltshire (CW) accuracy relies on the separation which is relative to its orbital radius rather than just separation alone. Furthermore, the linearisation error also grows with propagation time (Karlgaard and Lutze 62). Additionally, rectilinear LVLH coordinates can represent the large along-track offsets poorly, which is in contrast to the curvilinear formulations that aid in reducing much of this geometric error (Sullivan et al. 2; Willis et al. 14).

So a controlled response also depends on its initial state, where zero relative velocity creates bounded along track and cross track offsets. Radial offsets place the chaser on a different orbit with its respective period and energy, creating along-track drift. This suggests that a ‘universal failure distance’ for CW should not be expected.

2.5. Proportional-Derivative (PD) Control Design and Closed-Loop Dynamics

The PD controller acted on the LVLH relative state, which has an origin defaulted at zero relative velocity. For its relative position and velocity ρ and $\dot{\rho}$, and desired state ρ_d and $\dot{\rho}_d$ the acceleration is represented best by:

$$a = -K_p(\rho - \rho_d) - K_d(\dot{\rho} - \dot{\rho}_d), \quad (6)$$

subject to a magnitude saturation: if $\|a\|$ exceeds a maximum a_{max} , the command is scaled back to a_{max} , preserving its direction. The runs reported here use scalar isotropic gains where acceleration was limited $a_{max} = 0.05 \text{ m/s}^2$. Commands above this limit were scaled to a_{max} without having their direction changed and this limit was treated as a study parameter instead of a specific thrust value. Varying a_{max} helped test its influence separately.

For the assumed 100 kg chaser, the baseline acceleration cap of 0.05 m/s^2 corresponds to the maximum thrust of 5 N. This lies within the thrust range of small-spacecraft chemical propulsion systems, but represents a relatively high manoeuvring authority compared with low-thrust electric-propulsion systems (“4.0 In-Space Propulsion”). So, the kilometre-scale crossing distances reported for the baseline case must not be transferred directly to such missions. Thrust-cap sensitivity in Section 3.7 was therefore used in order to quantify how available control authority had affected the reported crossings.

The same control law and gains were then applied independently to the CW and nonlinear models. CW command used the current CW state while the nonlinear command used the relative state; it was reconstructed from the separately propagated inertial states of the chaser and target. This command was then rotated to the inertial frame from the LVLH frame before it was applied to the nonlinear chaser which allowed both models to then use the same controller while still responding to their own propagated states.

The gains are set by treating each axis as an independent second-order system and applying the critical-damping rule:

$$K_p = \omega_n^2, \quad (7a)$$

$$K_d = 2\omega_n, \quad (7b)$$

with $\omega_n = 6n$. This value was chosen to set the nominal controller bandwidth to be above the orbital rate. So for $n \approx 1.13 \times 10^{-3} \text{ rad/s}$, the selected gains were $K_p \approx 4.61 \times 10^{-5} \text{ s}^{-2}$ and $K_d \approx 1.36 \times 10^{-2} \text{ s}^{-1}$. Effectively following a standard second-order feedback design (Schaub and Junkins). However, this rule is exact only for a decoupled double integrator. The in-plane CW plant

carries Coriolis coupling and a radial anti-stiffness, and the cross-track axis carries a restoring stiffness, so the realised closed loop is not exactly critically damped. Meaning that the resulting closed-loop modes were calculated using the state matrix $A - BK$.

2.6. Force Models

The nonlinear reference uses a selectable force model, where its baseline is two body point mass gravity represented by:

$$a = \frac{-\mu r}{\|r\|^3} . \quad (8)$$

Two perturbations can be added. The first is the second zonal harmonic J_2 , the dominant non-spherical gravity perturbation in low Earth orbit (Schweighart 12), in the standard zonal form:

$$a_{J_2} = -\frac{3}{2}J_2 \frac{\mu R_\oplus^2}{r^5} \left[x \left(1 - \frac{5z^2}{r^2} \right), y \left(1 - \frac{5z^2}{r^2} \right), z \left(3 - \frac{5z^2}{r^2} \right) \right], \quad (9)$$

with $R_\oplus$ the Earth equatorial radius and $r = \|r\|$. The second is atmospheric drag,

$$a_{drag} = -\frac{1}{2}\rho \frac{C_d A}{m} \|v_{rel}\| v_{rel} , \text{ where } v_{rel} = v - \omega_\oplus \times r \quad (10)$$

is the velocity relative to the co-rotating atmosphere. This is the standard quadratic (velocity squared) drag model which includes the exponential-density parameters ρ_0 and H . They are selected as study inputs in Table 2. These terms define these four configurations: two-body, two-body using J_2 , two-body using drag, and then two-body with both perturbations. In the main comparison, two-body is basically used as a reference in order for the measured discrepancy in trajectories between CW and nonlinear and this helps to reflect the linearisation error under the stated assumptions. Separately, J_2 and drag are used in the supporting studies found in Sections 3.5 in order to calculate the mismatch that is added on by forces which are excluded from the CW model.

2.7. Numerical Propagation and Reference Scenario

The CW and nonlinear models were both integrated using the same fixed step fourth order Runge-Kutta (RK4) scheme. CW branch integrated the linear relative state equations while the nonlinear branch focused on the inertial states based on the target and chaser separately, after which their relative motion was reconstructed in the LVLH frame.

Different equations still end up producing different truncation errors albeit the same integration scheme was used for both and so their truncation errors were not assumed to cancel. Through step-refinement and two-body invariant tests, the numerical accuracy of the propagators was able to be assessed.

The reference orbit here is a circular 400 km Low Earth Orbit with an orbital radius that is about 6778 km; an inclination of 51.6° is used with a propagation horizon of one period. This orbit has a mean motion of $n \sim 1.13 \times 10^{-3} \text{ rad/s}$ and a period of $T \sim 5554 \text{ s}$. Based on the convergence studies, 4001 samples per orbit corresponded to a time step of approximately 1.39 s. The separation sweep instead used 2001 samples per orbit that corresponded to approximately 2.78 seconds. It was these resolutions which were selected using the convergence study that is reported in Section 3.1. Also note that the chaser's inertial state was reconstructed from the LVLH relative state.

For the rest of the propagation horizon, the sustained arrival is defined to have remained at within 1 m of the LVLH origin. Here, it was used more so as a position-based criterion and did not include an approach corridor, relative attitude, attitude-rate, capture constraints or contact. Therefore, the outcome was reported as a sustained position arrival rather than docking.

The modelling framework was implemented as a custom Python package where separate modules handled the CW and nonlinear dynamics, LVLH transformations, RK4 integration, PD control, force models and the comparison metrics. Table 2 is used to illustrate the parameters and constants which were used in the simulations.

Table 2. Implemented constants and scenario parameters (SI units).

Parameter	Symbol	Value
Earth gravitational parameter	μ	$3.986 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$
Earth equatorial radius	$R_\oplus$	$6.378 \times 10^6 \text{ m}$
Second zonal harmonic	J_2	1.083×10^{-3}
Earth rotation rate	$\omega_\oplus$	$7.292 \times 10^{-5} \text{ rad s}^{-1}$
Reference orbit altitude		400 km (radius $6.778 \times 10^6 \text{ m}$)
Reference orbit inclination	i	51.6° (perturbed reference)
Mean motion	n	$1.13 \times 10^{-3} \text{ rad s}^{-1}$
Orbital period	T	5,554 s
Integrator		fixed-step classic RK4
Samples per orbit		4001; 2001
Propagation horizon		One orbital period

Controller natural frequency	ω_n	6n
Proportional, derivative gain	K_p, K_d	$4.608 \times 10^{-5} s^{-2},$ $1.358 \times 10^{-2} s^{-1}$
Control saturation	a_{max}	$0.05 m s^{-2}$
Position tolerance		1.00 m
Drag coefficient, area, mass	C_d, A, m	2.20, $1.00 m^2$, 100 kg
Atmospheric density model	ρ	$\rho_0 = 3.90 \times 10^{-12} kg m^{-3}$
Scale height		60.0 km

2.8. Experiments and Comparison Metric

The central experiment was the separation sweep seen in Section 3.6, which measured how the controlled CW and nonlinear trajectories diverged as initial separation and offset direction varied. For each model, the controlled trajectory was created by applying the same PD law and gains while ensuring that each controller responds to its model's current state. The primary metric was the maximum Euclidean separation between the two relative trajectories which occur over one orbit, best represented by:

$$d_{max} = \max_t \| \rho_{CW}(t) - \rho_{ref}(t) \| \quad (11)$$

The propagation and control implementations were checked analytically and numerically before this difference was treated as a consequence of the model dynamics. Alongside this, NASA's GMAT was used as a reference for comparison with the nonlinear model. To confirm the expected closed-loop behaviour an illustrative rendezvous case was also used. In terms of sensitivity, separate studies were utilised to vary the following: controller gains, divergence criterion, thrust limit, perturbation model, radial initial state, controller bandwidth and propagation horizon. If the controllers saturate differently, the difference between their thrust-limited responses was also captured. Therefore, performance was assessed using the time to sustained arrival and the terminal position and velocity errors, where the control effort was reported separately as the cumulative velocity increment, seen from:

$$\Delta v = \int \| a_c(t) \| dt \quad (12)$$

2.9. Validation Process

Analytical, numerical and external checks were used to evaluate that the reported trajectory differences did not arise from numerical or implementation errors but mainly from the model dynamics. To check the CW propagator independently, it was reviewed closely against the corresponding closed-form

state-transition solution. For the unsaturated linear closed loop, the exact solution originated from the matrix exponential of the closed loop state matrix itself $A - BK$. In terms of physical and numerical checks, the two-body propagator is examined through the conservation of angular momentum and specific energy. Furthermore, through a comparison with the analytical orbital period - a convergence study managed the required time-step resolution.

The nonlinear reference was then compared with NASA's General Mission Analysis Tool (GMAT), by supplying the same reference scenario. Note that GMAT's force and propagation models are documented independently (Hughes, Mathematical Specifications 56-61; Hughes et al., "Verification and Validation"), and that the software has previously also been compared with independent astrodynamic tools that include: STK, FreeFlyer and MATLAB (Hughes et al., "Verification and Validation" 5). To match the simulator, a custom potential file had to be supplied alongside the simulator values and a single zonal coefficient $C_{20} = -J_2/\sqrt{5}$. Since GMAT prefers to represent the exponential atmosphere via a user-supplied table as opposed to a direct density equation, this meant that a one-record table was required in order to reproduce the simulator's density law. Additionally, a spherical Earth definition was used so that both tools had consistently calculated altitude. J2000 was chosen as the epoch, in order to reduce the difference between the simulator's pole representation and GMAT's MJ2000Eq frame; nutation was non-zero at this epoch, hence a residual frame discrepancy still remained.

In this comparison, the implementations directly tested are as follows: nonlinear free propagation, LVLH reconstruction and relative motion under two-body, J_2 and drag force models, and inertial-state initialisation. It did not test the PD controller, LVLH-to-inertial transformation command, nor the closed-loop implementation; those components were assessed separately instead through the numerical and analytical tests seen in the upcoming section.

3. RESULTS AND DISCUSSION

The CW and nonlinear rendezvous models were first implemented in a custom Python framework, after which their numerical behaviour and closed-loop agreement under the reference scenario were evaluated.

3.1. Convergence and Implementation Verification

To ensure numerical stability before conducting extensive limit studies, a time-step convergence analysis was performed (Table 3, Figure 1). Using a closed-loop metric and the Clohessy-Wiltshire (CW) integrator difference, the CW integration error was found to decrease at the expected fourth-order RK4 rate before reaching a floating-point floor. For a 1 km along-track case, the system converged at 2,001 samples per orbit, with the metric remaining within 0.01%.

The numerical CW propagation agreed with the analytical free-drift solution within 2.02×10^{-9} m over one orbit. Similarly, the unsaturated controlled RK4 solution matched the exact matrix exponential

solution within 3.02×10^{-8} m at 4,001 samples per orbit, maintaining fourth-order convergence. Further verification of the two-body propagator (Table 4) showed that relative changes in specific angular momentum and energy remained at the 10^{-14} level, and the propagated orbital period matched the analytical value within 10^{-13} , confirming the propagator's implementation.

The software implementation was further validated through 53 automated tests across nine modules. Key checks included:

- **Frame Transformations:** Control commands were rotated from the LVLH to the inertial frame and back with an error of approximately 10^{-13} .
- **Command Constraints:** Command saturation limits were verified without altering vector direction.
- **Controller Logic:** The PD controller correctly returned zero acceleration at the desired state.
- **Nonlinear Integration:** A central finite difference estimate of the nonlinear chaser acceleration confirmed that the control term was correctly applied to the nonlinear equations with a relative error below 10^{-6} .

These checks confirmed that the numerical and implementation errors were in fact relatively small compared to the trajectory differences examined further on.

Table 3. Time-step convergence of CW integrator and closed loop divergence metric.

Samples / orbit	Step Δt [s]	CW error [m]	Divergence [mm]	Relative change
251	22.21	1.25×10^{-4}	2.795	23.5%
501	11.11	7.83×10^{-6}	2.208	2.4%
1,001	5.554	4.90×10^{-7}	2.259	0.15%
2,001	2.777	3.02×10^{-8}	2.263	0.0093%
4,001	1.388	2.02×10^{-9}	2.263	0.0010%
8,001	0.694	5.54×10^{-10}	2.263	0.0001%
16,001	0.347	5.26×10^{-10}	2.263	reference

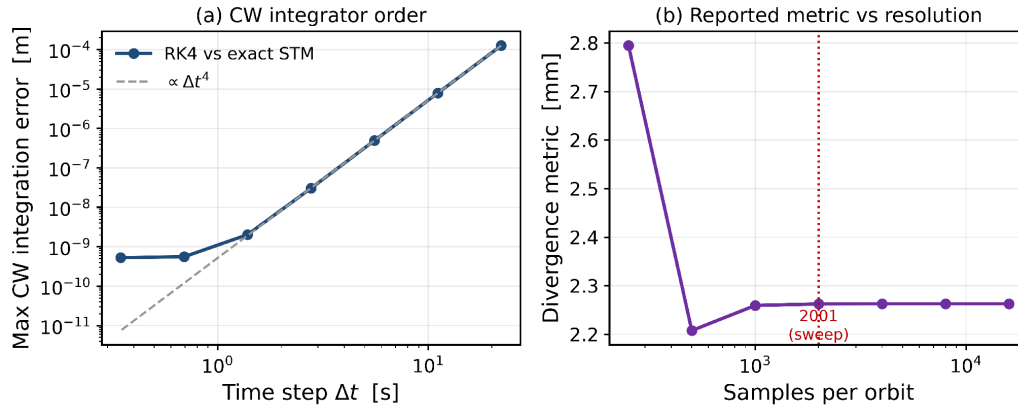


Figure 1. Time-step convergence of the CW integrator and closed-loop divergence metric.

Table 4. Two-body propagator conservation and orbital-period agreement over two orbits.

Quantity	Value
Maximum relative energy drift	1.37×10^{-14}
Maximum relative angular-momentum drift	6.61×10^{-15}
Measured period	5,554 s
Analytic period	5,554 s
Period relative error	1.34×10^{-13}

3.2. External Comparison with GMAT

To validate the nonlinear propagator, the model was compared against NASA’s GMAT across five initial geometries: three 1 km axis-aligned offsets, one 3D mixed offset, and one 5 km radial offset as shown in Table 5 and Figure 2. Both tools utilized a one-orbit horizon and a fixed time step of 1.3884 seconds. Under two-body gravity, the relative-motion difference remained below 1 μm for all cases, confirming the consistency of the nonlinear propagation, LVLH frame transformations, and initial-state reconstruction.

When J_2 perturbations were included, the absolute position residual reached 2.58 m, while the relative-motion residual remained at the millimeter level. This discrepancy in absolute position is attributed to differing J_2 frame definitions: the simulator used a fixed inertial symmetry axis, whereas GMAT used an Earth-fixed frame following the true pole (Hughes, Mathematical Specifications 58). Because this perturbation acted similarly on both spacecraft, the error largely cancelled out when calculating the relative state.

The smallest CW-to-reference trajectory difference was approximately 12 μm (at a 50 m offset), which is roughly five orders of magnitude smaller than the 1 m study criterion. This confirms that the nonlinear reference and state reconstruction are accurate to a level far exceeding the requirements of this study. While this comparison did not test the PD controller or closed-loop implementation, it establishes a high-fidelity nonlinear reference. Consequently, with the nonlinear reference having been established for the tested cases, the selected controller was characterised before performing the closed-loop comparisons.

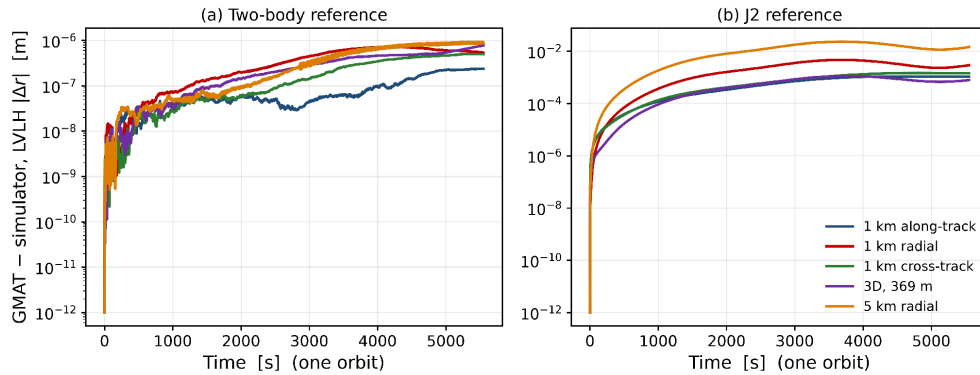


Figure 2. Relative LVLH differences between GMAT and the nonlinear model across five initial geometries.

Table 5. Relative-motion differences between GMAT and the nonlinear model across five initial geometries.

Case	LVLH (m)	offset $\ \rho_0\ $ (m)	2-body (m)	max (m)	2-body (m)	RMS J_2 max (m)	J_2 (m)	RMS
Along-track	(0, 1000, 0)	1000	2.47×10^{-7}	1.1×10^{-7}	1.08×10^{-3}	7.3×10^{-4}		
Radial	(1000, 0, 0)	1000	7.2×10^{-7}	4.5×10^{-7}	4.65×10^{-3}	2.7×10^{-3}		
Cross-track	(0, 0, 1000)	1000	5.0×10^{-7}	2.6×10^{-7}	1.47×10^{-3}	9.4×10^{-4}		
3D mixed	(200, -300, 80)	369	7.8×10^{-7}	3.5×10^{-7}	1.12×10^{-3}	6.8×10^{-4}		
Radial (5 km)	(5000, 0, 0)	5000	9.4×10^{-7}	4.9×10^{-7}	2.33×10^{-2}	1.36×10^{-1}		

3.3. Controller Gain Selection and Closed-Loop Characterisation

Before using the PD controller in the model-comparison studies, sixteen gain combinations were tested on the unsaturated CW model in order to identify a stable reference response. Thirteen gain pairs managed to reach the 1 m tolerance within just one orbit, while the remaining three were weakly damped and did not

converge within that time frame. Amongst the converging cases, higher gains generally reduced the arrival time, but raised the Δv required in exchange (Figure 3). The grid point nearest to the $\omega_n = 6n$ design utilised $K_p = 4.6 \times 10^{-5}$ and $K_d = 1.36 \times 10^{-2}$. It reached the tolerance where $\Delta v \approx 2.32 \text{ m s}^{-1}$ in approximately 1598 seconds. This placed the design point near the lower-control-effort end of the converging cases, but in doing so had supported retaining the exact design gains as the baseline controller.

The selected baseline gains had produced stable closed-loop modes with damping ratios between 0.929 and 0.986, meaning that the implemented CW closed loop had to therefore be near-critically damped and not exactly critically damped. Cross-track mode was then decoupled and the Coriolis coupling had divided the in-plane response between fast and slow modes. The slow in-plane mode produced the longest settling time (Table 6). Thus, the selected gains were then used as the baseline controller for the representative rendezvous case.

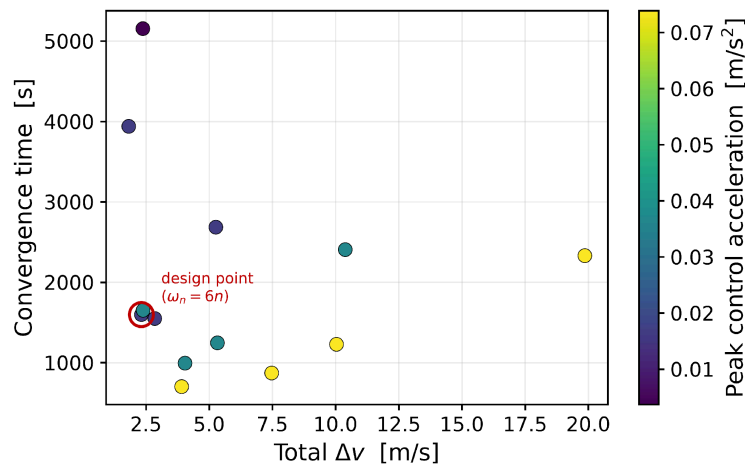


Figure 3. Convergence time, total Δv and peak acceleration across the PD gain sweep.

Table 6. Modal properties of the PD-controlled CW model.

Mode	Real (s^{-1})	part Imaginary (s^{-1})	part $ \lambda / n$	Damping ratio ζ	2% settling time
In-plane, fast	-9.61×10^{-3}	$\pm 3.84 \times 10^{-3}$	9.15	0.929	416 s (0.075 orbit)
Cross-track	-6.79×10^{-3}	$\pm 1.13 \times 10^{-3}$	6.08	0.986	589 s (0.106 orbit)
In-plane, slow (dominant)	-3.97×10^{-3}	$\pm 1.56 \times 10^{-3}$	3.77	0.931	1008 s (0.181 orbit)

3.4. PD-Controlled Rendezvous

Using the selected gains, a representative 369 m rendezvous was propagated in order to establish the baseline closed-loop response prior to comparing the CW and nonlinear models. The in-plane trajectory curved towards the target, which was consistent with the Coriolis coupling and the position error fell below the 1 m tolerance after about 1591 seconds (0.29 orbit). Peak control acceleration was approximately 0.017 m s^{-2} , which remained below the acceleration cap. The final position and velocity errors were about $1.50 \times 10^{-7} \text{ m}$ and $6.62 \times 10^{-10} \text{ m s}^{-1}$, respectively, while the total Δv was 2.33 m s^{-1} (Figure 4; Table 7).

An approach corridor, relative attitude, attitude-rate, contact or any capture constraints were not included in the model. Outcomes were therefore treated as sustained position arrivals instead of completed dockings. This case established the baseline response of the selected controller before investigating the effect of perturbations omitted from the CW model.

Table 7. Performance metrics for the representative CW rendezvous.

Metric	Value
Initial position error	369.3 m
Final position error	$1.50 \times 10^{-7} \text{ m}$
Final velocity error	$6.62 \times 10^{-10} \text{ m s}^{-1}$
Convergence time (below 1 m, sustained)	1,591 s (≈ 0.29 orbit)
Peak control acceleration	0.017 m s^{-2} (unsaturated)
Total Δv	2.33 m s^{-1}

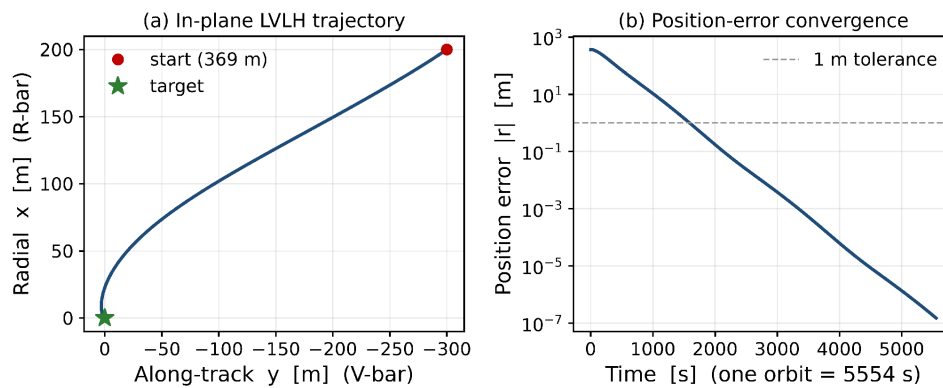


Figure 4. Representative PD-controlled rendezvous on the CW model.

3.5. Effect of Omitted Perturbations

The next comparison tested if the forces omitted from the baseline CW model had introduced a significant additional mismatch. Using a 1 km along-track free-drift scenario, J_2 increased the maximum position difference from 2.78 to 5.81 m, while the final difference remained approximately 2.78 m. This indicated that its larger effect occurred predominantly during the propagated trajectory rather than as a larger terminal discrepancy. Drag only managed to change it by about 4 mm (Table 8). As the target and chaser used the same drag-area to mass ratio $C_d A/m$, most of the drag acceleration acted nearly equally on both vehicles and largely cancelled when the difference between their states was taken. The only remaining component was the difference between the accelerations acting upon the two vehicles, and so this affected the relative trajectory, with J_2 having produced the larger differential effect among the perturbations that were tested in this case (Schweighart 12).

While utilising closed-loop control, the corresponding 369 m case had provided a difference of approximately 1.7 cm early on in the approach, and as both trajectories converged onto the target, the difference later fell to about 3×10^{-9} m (Figure 5). In this case, feedback repeatedly corrected the relative state whilst keeping the resulting difference between the trajectories small in comparison to the free-drift discrepancy. The two-body reference was therefore retained for the central separation sweep to ensure that the measured difference primarily reflected the CW linearisation error and kept the effects of J_2 and drag to be tested separately to avoid any additional mismatch.

Table 8. Open-loop CW-to-reference difference under tested force models.

Force model	Maximum position error [m]	Final position error [m]
two-body	2.781	2.781
two-body with J_2	5.813	2.779
two-body with drag	2.785	2.785
two-body with J_2 and drag	5.816	2.784

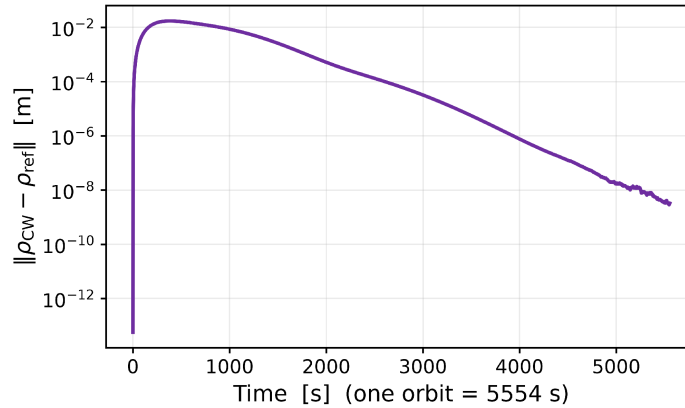


Figure 5. Controlled CW-to-reference trajectory difference for the rendezvous.

3.6. Closed-Loop Agreement Across Separation

As the two-body reference was established, the central experiment had varied the initial separation range from 50 m to 50 km along the pure radial, along-track and cross-track offsets with zero initial relative velocity over one orbit. The same PD law and gains were also applied separately to the CW model and two-body nonlinear models. In the closed-loop case the maximum difference grew from the micrometre scale at a 50 m offset to a few millimetres near 1 km (Figure 6). So, continuous feedback repeatedly corrected the evolving state, and kept the trajectories more closely aligned compared to the free-drift across the close-range cases.

The difference between the open and closed-loop behaviors had become clearer as the initial separation was increased (Figure 7). By using the 1 m trajectory difference criterion, the free drift crossing occurred at approximately 95 m radially, 600 m along-track and 850 m cross-track. Under PD control, the very same threshold moved to approximately 5.5 km radially, 7.7 km along-track and 10 km cross-track (Table 9). The radial agreement range increased by a factor of approximately 57, compared with approximately 13 for along-track and 12 for cross-track directions. Across the tested cases, the directional ordering remains unchanged, meaning that radial offsets produce the earliest divergence, followed by the along-track and then finally the cross-track offsets. Here, the first non-arrival refers to the first tested separation range where the nonlinear reference had no longer satisfied the sustained 1 m arrival criterion within the horizon. The first non-arrival was observed between approximately 8 and 14 km for radial offsets and 14 and 27 km for along-track offsets, all the while no cross-track non-arrival had been observed within the tested range up to 50 km (Table 9).

Under the tested initial conditions where relative velocity is zero, the radial case was not energy-matched to the target orbit. This placed the chaser on an orbit with a different period that subsequently accumulated an along-track drift which the controller had to remove. For the radial case, the first non-arrival had occurred when the nonlinear reference no longer satisfied the arrival condition within one

orbit, whilst the CW model had still continued to predict arrival. Consequently, the resulting trajectory difference then rose to tens of kilometres. This metric represented both model disagreement and a different closed-loop outcome rather than CW linearisation error alone. The radial case had therefore lost its agreement first under the zero-relative-velocity initial condition that had been tested, indicating that the crossing was dependent on more than just separation.

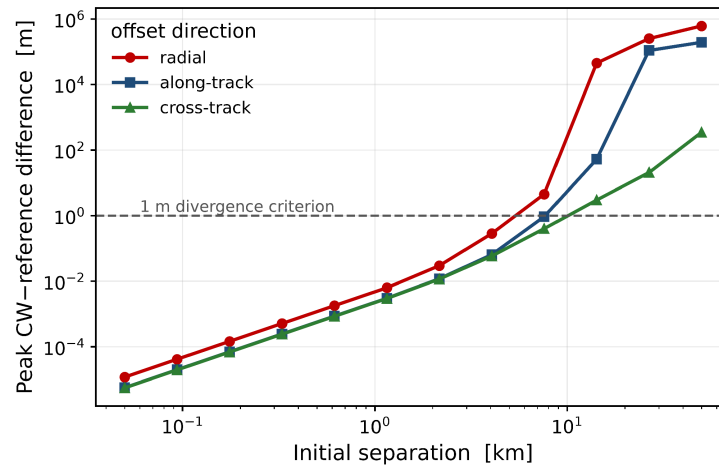


Figure 6. Closed-loop CW-to-reference difference across initial separation.

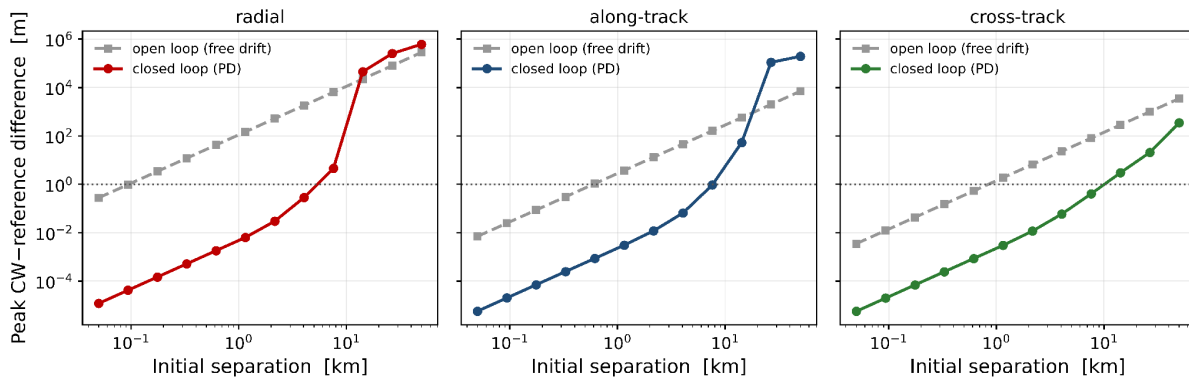


Figure 7. Open and closed-loop CW-to-reference differences across initial separation.

Table 9. Estimated 1 m crossings for the three initial-offset directions.

Offset direction	Open-loop separation	Closed-loop separation	Feedback extension	First non-arrival
radial	≈ 95 m	≈ 5.5 km	≈ 57×	8 to 14 km
along-track	≈ 600 m	≈ 7.7 km	≈ 13×	14 to 27 km

cross-track	≈ 850 m	≈ 10 km	$\approx 12\times$	not observed up to 50 km
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***Note:** Feedback extension was calculated as the closed-loop crossing separation that was divided by the corresponding open-loop crossing separation.*

3.7. Sensitivity of the Closed-Loop Agreement Crossings

To determine whether the reported crossings were tailored to the radial state and selected comparison criterion, sensitivity analysis had varied the error threshold, radial energy matching as well as available thrust. When the absolute limit was reduced from 1 m to 0.1 m, the radial crossing moved from 5.5 to 3.0 km, along-track crossing from 7.7 to 4.5 km and the cross-track crossing from 10 to 4.8 km. This was all the while the 1% relative criterion had moved the radial and along-track crossings outward to 9.3 and 16 km respectively (Table 10), whilst the cross-track difference never managed to reach the 1% within the 50 km sweep. The directional order of radial, along-track and cross-track remained unchanged across the three tested criteria and only the exact numerical values had changed.

Table 10. Closed-loop crossing separations under the three criteria.

Criterion	Radial	Along-track	Cross-track
0.1 m absolute	3.0 km	4.5 km	4.8 km
1.0 m absolute	5.5 km	7.7 km	10 km
1% of separation	9.3 km	16 km	n/a

Poorer radial behaviour here was then tested using an energy-matched initial state $\dot{y}_0 = -2nx_0$, as opposed to the zero-relative-velocity (Figure 8). In free drift, the zero-velocity state produced approximately 79 times more along-track drift than when using an energy-matched case without control. Under control, the two cases had produced similar differences at close range, but the energy matching moved the radial 1 m crossing from about 5.5 to 6.0 km. So at a 12 km separation, the peak trajectory difference was reduced from 122 to 16 m. These results therefore support an association between the energy mismatching and the poorer radial behaviour of the zero-velocity initial state.

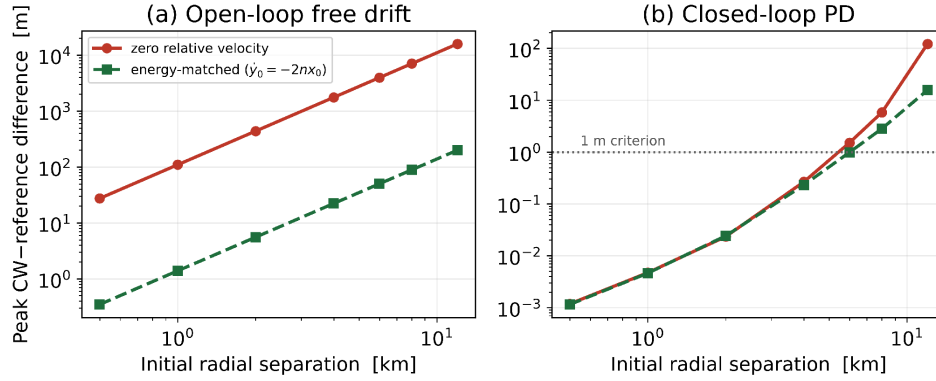


Figure 8. Effect of radial energy matching on free-drift and controlled trajectory differences.

Finally, the acceleration cap was varied to examine the influence of available control authority and so, the radial 1 m crossing moved from approximately 2.0 km at 0.01 m s^{-2} to 5.5 km at the baseline cap of 0.05 m s^{-2} , and to 7.9 km at 0.10 m s^{-2} (Table 11). This greater thrust authority allowed the controller to remove the radial-case drift across a larger initial separation. Together, these sensitivities showed that the reported onset crossings were relying on the comparison criterion, available control authority and initial radial state instead of specifically displaying a universal CW boundary. It was this that motivated the final robustness check against controller bandwidth and propagation horizon.

Table 11. Radial 1 m crossing separation across the tested thrust caps.

Thrust cap a_{max}	Radial 1 m onset
0.01 m s^{-2}	$\approx 2.0 \text{ km}$
0.05 m s^{-2}	$\approx 5.5 \text{ km}$
0.10 m s^{-2}	$\approx 7.9 \text{ km}$

3.8. Robustness to Controller Bandwidth and Propagation Horizon

To test whether the directional pattern had remained robust to the changes in controller and propagation period, the controller bandwidth was varied at a one orbit horizon and the propagation horizon was varied using the acceleration cap and baseline controller (Tables 12a and 12b). Across the tested bandwidths and horizons, the radial crossing had remained the earliest, followed by the along-track and cross-track crossings.

By varying the propagation horizon from 0.5 to 2 orbits, the reported crossings were not altered to the reported precision, with values remaining at approximately 5.4 km radially, 7.7 km along-track and 10.1

km cross track (Table 12b). So, the crossing was insensitive to propagation duration across this tested range.

The controller bandwidth produced a measurable change in the crossing values (Table 12a). Increasing ω_n / n from 4 to 8 had moved the radial crossing from 5.9 to 5.2 km, along-track crossing from 9.0 to 7.0 km and the cross-track crossing from 10.7 to 9.5 km. Despite these changes, the directional ordering remained unchanged across the tested grid.

Table 12 (a). Controller-bandwidth (ω_n / n) sensitivity at the 1-orbit horizon.

ω_n / n	Radial	Along-track	Cross-track
4	≈ 5.9 km	≈ 9.0 km	≈ 10.7 km
6 (baseline)	≈ 5.4 km	≈ 7.7 km	≈ 10.1 km
8	≈ 5.2 km	≈ 7.0 km	≈ 9.5 km

Table 12 (b). Propagation-horizon sensitivity at the baseline controller.

Horizon	Radial	Along-track	Cross-track
0.5 orbit	≈ 5.4 km	≈ 7.7 km	≈ 10.1 km
1 orbit	≈ 5.4 km	≈ 7.7 km	≈ 10.1 km
2 orbits	≈ 5.4 km	≈ 7.7 km	≈ 10.1 km

***Note:** The baseline robustness values differ slightly from the refined crossings used in Table 9 as the sensitivity analysis used the robustness sweep grid.*

4. CONCLUSION

Within the tested scenario, the results supported the hypothesis that closed-loop correction kept the CW and nonlinear trajectories within agreement over greater initial separations compared to that of free propagation. So, using the same saturated PD law applied independently to both models under the tested circular Low Earth Orbit scenario, the 1 m peak-difference threshold had occurred at approximately 5.5 km for a radial offset, 7.7 km for an along-track offset and at 10 km for a cross-track offset. While in free drift, the exact same threshold was attained at values of approximately 95 m, 600 m and 850 m, respectively. The solution to the proposed research question was direction-dependent meaning that disagreement generally increased alongside initial separation, however the feedback delayed that growth by limiting the accumulation of linearisation error at closer separations. This was done via repeatedly reducing the relative state.

The crossings did not all represent a pure loss of CW linearisation accuracy. At closer separations, this was evident from the small difference between the trajectories which suggested that feedback effectively limited the linearisation error, when looking at closer ranges. Nevertheless, at the radial crossing, the

nonlinear trajectory began to differ in outcome away from the CW prediction. A radial offset that was released from zero relative velocity was not energy-matched to the target orbit, and so ended up developing an along-track drift. Beyond the radial crossing, the difference was no longer just a small error between two trajectories as the nonlinear case had developed along-track drift. The moment that the available thrust was insufficient to correct it within one orbit, it became insufficient in satisfying the arrival condition, though CW still managed to predict the convergence. The radial test that was energy-matched had delayed this split, and so the thrust-cap study moved the 1 m crossing from approximately 2.0 km to 7.9 km. This therefore showed that the crossing depended on the initial state and available control authority, as well as the controller, propagation time and selected error criterion. Hence, the reported crossing should not be interpreted as a universal limit of the CW equations.

Within the orbit and controller conditions that were tested, CW was thereby useful for the preliminary closed-loop analysis and the models remained within the selected 1 m agreement criterion. Nonlinear verification became more important once thrust saturation, an energy-mismatched initial state or forces excluded from CW had affected the outcome.

A two-body reference was deliberately used in the main separation to examine the CW linearisation error on its own. The kilometre-scale crossings should have therefore not been read to include the J_2 operating ranges. This is within the separate perturbation cases where the differential J_2 was already larger than the close-range two-body residual. The nonlinear propagator agreed with GMAT to below one micrometre for the tested two-body cases, which in turn supported its use as the reference model. That exact validation was limited to the selected geometries and a one-orbit horizon.

Further investigations could look into repeating the comparison for other circular-orbit altitudes and inclinations, eccentric reference motion, unequal ballistic coefficients, different controller gains and thrust limits, and longer propagation periods. Furthermore, navigation and actuator errors may also be introduced. Future work could also extend the analysis from rendezvous to docking and would additionally require relative-velocity limits, approach corridors, attitude dynamics and capture constraints.

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