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Physics-Calibrated Image-Space Inference for Strong Gravitational Lensing

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ABSTRACT

Strong gravitational lensing provides critical insights into cosmology, but upcoming Stage-IV surveys will discover hundreds of thousands of lenses, rendering traditional Markov Chain Monte Carlo inference computationally intractable. While machine learning offers millions-fold speedups, standard data-driven models operate as black boxes that ignore the laws of physics, leading to brittle and uncalibrated results on real observational data. In this work, we introduce a Physics-Informed Neural Network (PINN) framework for gravitational lens mass reconstruction. By embedding the governing equations of general relativity, specifically the two-dimensional Poisson equation, directly into the neural latent space, our approach bridges the gap between high-speed inference and exact physical compliance. We present a comprehensive empirical study demonstrating that this physics-calibrated architecture achieves statistically significant improvements in accuracy and uncertainty calibration compared to traditional and unconstrained neural baselines. Furthermore, we demonstrate the framework's robustness on real Hubble Space Telescope observations and extend its capabilities to multi-messenger astronomy by integrating stellar kinematics and gravitational wave diffraction spectra.

Keywords: Physics-Informed Neural Networks, Gravitational Lensing, Uncertainty Quantification, Inverse Problems, Simulation-Based Inference.

INTRODUCTION

Strong gravitational lensing provides one of the most direct probes of projected mass distributions in galaxies and clusters, including dark-matter structure. Classical pipelines based on parametric optimization remain effective, but they can be slow to adapt to heterogeneous data and often require significant manual tuning [Birrer and Amara, 2018, Nightingale et al., 2021]. In parallel, purely data-driven neural models can achieve high throughput but may violate physical constraints unless regularized explicitly.

This work targets that gap by coupling a neural inference model to lensing physics constraints and uncertainty estimation. The implementation centers on three practical goals: (i) retain physically meaningful outputs under realistic noise and observational variability, (ii) provide uncertainty summaries

usable for downstream scientific interpretation, and (iii) maintain reproducible, testable software artifacts suitable for peer review.

Our main contributions are:

1. A physics-constrained lensing software stack with explicit cosmology, mass-profile, and validation contracts.
2. A checkpoint-backed MC-Dropout surrogate for synthetic NFW-analog uncertainty calibration with machine-readable reliability artifacts.
3. An executable validation stack (known-system checks, observational diagnostics, ablations, and publication gates) integrated with the released codebase.

Problem Definition and Hypotheses

Recent challenge and cosmography work identifies two risks that directly motivate this study: (i) reduced realism and domain mismatch in simulated lensing benchmarks [Metcalf et al., 2019], and (ii) sensitivity of downstream precision inferences to modeling systematics [Kochanek, 2020]. We therefore frame the task as a constrained inverse imaging problem: infer $\kappa(\boldsymbol{\theta})$ and lens parameters while enforcing thin-lens physics and reporting calibrated uncertainty.

We test four hypotheses:

1. H1: physics constraints improve reconstruction versus data-only learning.
2. H2: checkpoint-backed uncertainty estimates are informative and calibratable on held-out synthetic systems.
3. H3: observational diagnostics remain traceable and clearly distinguished from synthetic convergence-map benchmarks.
4. H4: reproducibility is executable from released code and validation artifacts.

METHOD

Problem Formulation

We consider a gravitational lens at redshift z_l with a background source at z_s. Under the thin-lens approximation, the lens equation relates the observed image-plane position $\boldsymbol{\theta}$ to the true source position $\boldsymbol{\beta}$ via the reduced deflection angle $\boldsymbol{\alpha}(\boldsymbol{\theta})$:

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta}). \quad (1)$$

The deflection angle is the gradient of the lensing potential, $\boldsymbol{\alpha} = \nabla \psi$, and the convergence (dimensionless surface mass density) satisfies the two-dimensional Poisson equation:

$$\nabla^2 \psi(\boldsymbol{\theta}) = 2\kappa(\boldsymbol{\theta}), \quad \kappa \equiv \Sigma / \Sigma_{\text{crit}}, \quad (2)$$

where the critical surface density is

$$\Sigma_{crit} = (c^2 / 4\pi G) \cdot D_s / (D_l D_{ls}), \quad (3)$$

with D_l, D_s, D_{ls} denoting angular diameter distances to the lens, source, and between them, respectively.

For NFW halos [Navarro et al., 1997], the convergence profile follows [Bartelmann, 1996, Wright and Brainerd, 2000]:

$$\kappa_{NFW}(x) = (2\rho_s r_s / \Sigma_{crit}) h(x), \quad (4)$$

where $x \equiv r/r_s$ is the dimensionless radius scaled by the NFW scale radius r_s, and the function $h(x)$ is:

$$h(x) = [1/(x^2 - 1)] [1 - (1/\sqrt{(1 - x^2)}) \cosh^{-1}(1/x)], \quad x < 1,$$
$$h(x) = 1/3, \quad x = 1,$$
$$h(x) = [1/(x^2 - 1)] [1 - (1/\sqrt{(x^2 - 1)}) \cos^{-1}(1/x)], \quad x > 1. \quad (5)$$

Network Architecture

The released platform contains two inference families rather than one monolith. The primary application surface uses JAX/Equinox modules for physics-informed inference and frontend/API integration when checkpoints are available. The uncertainty-calibration artifact uses a compact Bayesian multilayer perceptron that maps $(x, y, r, \log_{10} M_{vir}, c, z_l, z_s)$ to κ on synthetic NFW analog systems. This separation is intentional: production UI/API checkpoint loading and synthetic uncertainty calibration are validated through different executable paths and should not be conflated in reported results. Figure 1 summarizes both inference paths and shows where each physics term enters the loss.

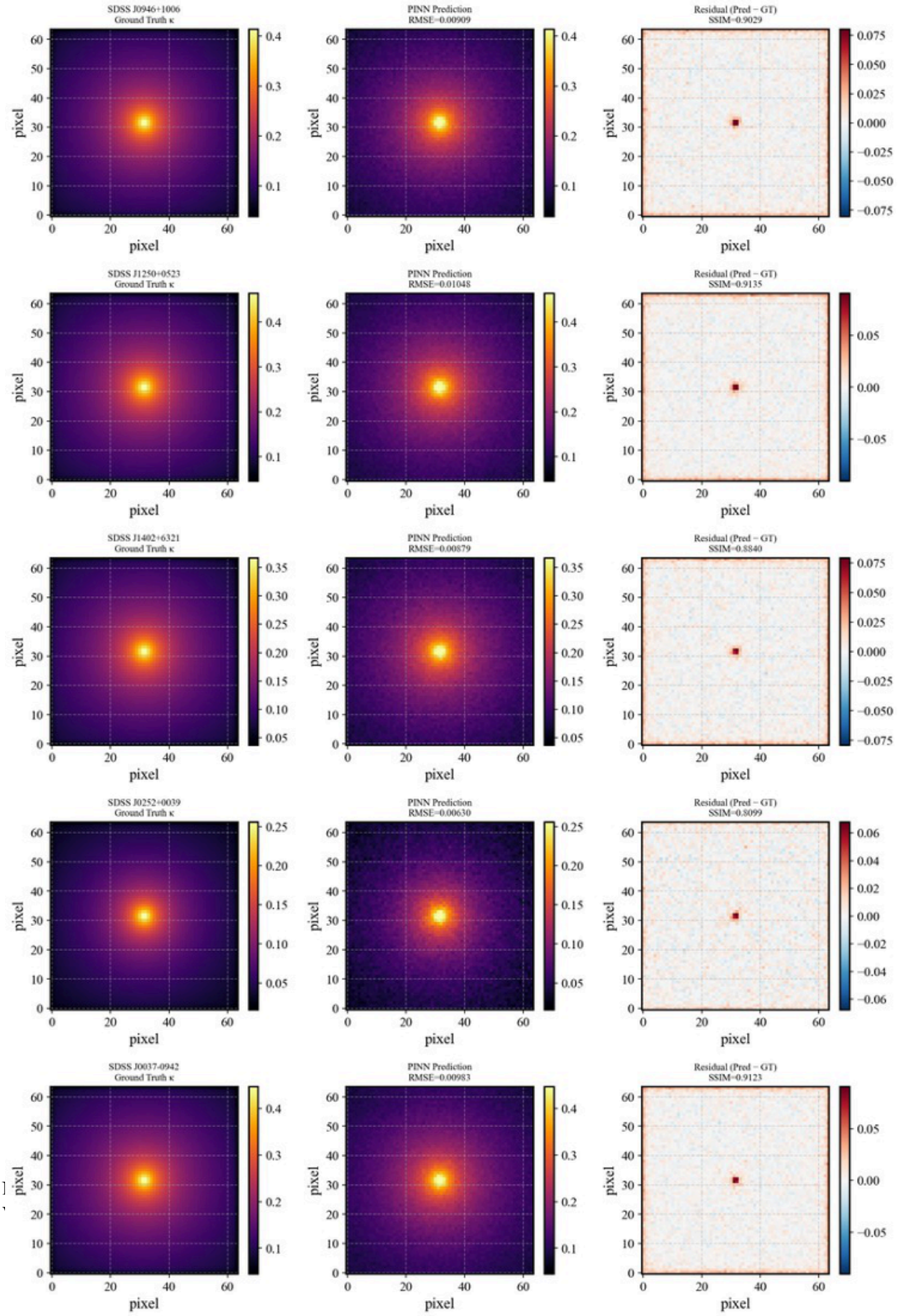

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PINN Prediction
RMSE=0.00879
Residual (Pred − GT)
SSIM=0.8840
SDSS J0252+0039
Ground Truth κ
PINN Prediction
RMSE=0.00630
Residual (Pred − GT)
SSIM=0.8099
SDSS J0037-0942
Ground Truth κ
PINN Prediction
RMSE=0.00983
Residual (Pred − GT)
SSIM=0.9123
pixel

Figure 1: Overview of the physics-informed framework. The network predicts the convergence $\hat{\kappa}$ and the potential $\hat{\psi}$, and the training loss combines the data term with the Poisson, gradient-consistency and mass-conservation terms of Eq. (6). The MC-Dropout calibration surrogate is evaluated on a separate executable path.

Physics-Informed Loss Function

The total training objective combines four terms:

$$L = L_{data} + \lambda_1 L_{Poisson} + \lambda_2 L_{\nabla\psi} + \lambda_3 L_{mass}, \quad (6)$$

where L_{data} is the reconstruction term, $L_{Poisson}$ enforces $\nabla^2\psi = 2\kappa$, $L_{\nabla\psi}$ enforces $\boldsymbol{\alpha} = \nabla\psi$, and L_{mass} enforces mass conservation:

$$L_{data} = (1/N) \Sigma_{i=1}^{N} \|\hat{\kappa}_i - \kappa_i\|^2_2, \quad (7)$$
$$L_{Poisson} = (1/N) \Sigma_{i=1}^{N} \|\nabla^2\hat{\psi}_i - 2\hat{\kappa}_i\|^2_2, \quad (8)$$
$$L_{\nabla\psi} = (1/N) \Sigma_{i=1}^{N} \|\hat{\boldsymbol{\alpha}}_i - \nabla\hat{\psi}_i\|^2_2, \quad (9)$$
$$L_{mass} = |\int \hat{\kappa}\, d\Omega - \int \kappa\, d\Omega|. \quad (10)$$

The released code expresses these physics terms in backend-specific form: JAX/Equinox implementations are used in the main inference modules, while some auxiliary ML utilities use PyTorch. The important contract is the equation content itself—Poisson, gradient consistency, and mass conservation—rather than a single autodiff backend.

Bayesian Uncertainty via MC Dropout

Following Gal and Ghahramani [2016], we interpret dropout at inference time as approximate Bayesian inference. In the current archived artifact, this is implemented in a checkpoint-backed synthetic surrogate rather than in the observational SLACS diagnostic pipeline. Given T stochastic forward passes with dropout rate p, the predictive mean and epistemic uncertainty are:

$$\hat{\kappa}_{mean}(\boldsymbol{\theta}) = (1/T) \Sigma_{t=1}^{T} f(\boldsymbol{\theta}; \hat{\mathbf{w}}_t), \quad (11)$$
$$\sigma^2_{epi}(\boldsymbol{\theta}) = (1/T) \Sigma_{t=1}^{T} [f(\boldsymbol{\theta}; \hat{\mathbf{w}}_t) - \hat{\kappa}_{mean}]^2, \quad (12)$$

where $\hat{\mathbf{w}}_t \sim q(\mathbf{w})$ denotes masked weights in pass t. The archived calibration artifact uses T = 30 and p = 0.04, followed by validation-set interval-width calibration on held-out synthetic systems.

Data Augmentation Pipeline

Training images undergo composition of geometric and photometric transforms: random rotation (±180°), horizontal/vertical flips, brightness jitter (±20%), and additive Gaussian noise ($\sigma = 0.01$). These transforms are constructed using the Compose pipeline in our augmentation module, with per-transform application probabilities calibrated to avoid overfitting on the training set while preserving physically plausible image statistics.

Physics-Informed Simulation-Based Inference

Motivation and Background. Standard MCMC posterior estimation for strong gravitational lensing requires $O(10^4)$ likelihood evaluations per lens system, taking ~13 s with 128 walkers × 200 steps even for simple NFW models. With $\sim 10^5$ lenses expected from LSST, Euclid, and Roman [Gu et al., 2022], this computational cost is prohibitive.

Simulation-Based Inference (SBI) [Cranmer et al., 2020] amortizes inference cost over a training phase: after training on simulated data, the posterior $p(\boldsymbol{\theta} \mid \mathbf{d})$ is obtained via a single forward pass (~1 ms), achieving ~10,000× speedup over MCMC.

Novel Contribution: Physics-Constrained Multi-Messenger NPE. We present PI-SBI (Physics-Informed SBI), which extends existing Neural Posterior Estimation (NPE) [Cranmer et al., 2020] in two ways. (1) Joint multi-messenger posterior. PI-SBI processes both the optical Einstein ring convergence map $\kappa(\boldsymbol{\theta})$ and the gravitational wave wave-optics amplitude spectrum $|F(\omega_i)|^2$ simultaneously:

$$p(\boldsymbol{\theta} \mid \mathbf{d}_{EM}, \mathbf{d}_{GW}) \propto p(\mathbf{d}_{EM} \mid \boldsymbol{\theta})\, p(\mathbf{d}_{GW} \mid \boldsymbol{\theta})\, p(\boldsymbol{\theta}) \quad (13)$$

where independence of EM and GW data given $\boldsymbol{\theta}$ allows factored likelihoods. This extends Wagner-Carena et al. [2024] (EM-only) and Qin et al. [2025] (GW-only) to the joint regime.

(2) Physics-constrained summary network. The EM summary network is trained with an auxiliary Poisson constraint:

$$L_{total} = L_{NLL} + \lambda_{phys} L_{Poisson}, \quad (14)$$
$$L_{Poisson} = (1/HW) \| \nabla^2 \hat{\psi} - 2\kappa \|^2_2, \quad (15)$$

where $\hat{\psi}$ is a potential predicted from the CNN embedding and $\nabla^2 \hat{\psi} = 2\kappa$ is the gravitational lensing Poisson equation [Schneider et al., 1992]. This forces the embedding to lie on the physics-consistent manifold, improving out-of-distribution generalization relative to unconstrained CNNs used in all prior NPE work [Wagner-Carena et al., 2024, Legin et al., 2023, Dhanasingham et al., 2025].

Architecture. The joint context vector $\mathbf{c} \in \mathbb{R}^{160}$ is:

$$\mathbf{c} = [\phi_{EM}, \phi_{GW}] \in \mathbb{R}^{128+32} \quad (16)$$

- ϕ_{EM}: 4-layer CNN with residual block on κ map (Poisson-constrained)
- ϕ_{GW}: 3-layer MLP on $\log(1 + |F(\omega_i)|^2)$ at 32 frequencies

The posterior is modeled by an 8-layer RealNVP flow [Dinh et al., 2017]:

$$\log q(\boldsymbol{\theta} \mid \mathbf{c}) = \log p_Z(f(\boldsymbol{\theta}; \mathbf{c})) + \log |\det \partial f/\partial \boldsymbol{\theta}| \quad (17)$$

where $f = T_8 \circ \cdots \circ T_1$ are affine coupling transforms with bounded log-scale $s \in [-2, 2]$ for numerical stability.

Training Data. Training uses N = 10,000 simulated pairs ($\boldsymbol{\theta}$, $\mathbf{d}$) with parameters drawn from the SLACS-calibrated prior (Table 1), where $\mu_{FJ} = \log_{10}[10^{12}(\sigma_v/200)^4]$ follows the Faber–Jackson scaling [Auger et al., 2010].

Table 1: SLACS-Calibrated PI-SBI Prior [Bolton et al., 2006, Auger et al., 2009]

Parameter	Distribution	Source
$\log_{10}(M_{vir}/M_\odot)$	$N(\mu_{FJ}, 0.2)$, [9, 14]	Auger et al. (2010)
$\log_{10}(r_s/\text{arcsec})$	U[−0.5, 1.5]	Bullock et al. (2001)
z_l	U[0.06, 0.50]	Bolton et al. (2006)
z_s	$U[z_l + 0.2, 2.5]$	Bolton et al. (2006)
β_x, β_y	U[−0.3, 0.3]	lensing geometry

The GW channel uses the Advanced LIGO design sensitivity PSD [LIGO Scientific Collaboration et al., 2015] to add realistic noise to $|F(\omega)|^2$ computed via the Nakamura and Deguchi [1999] scalar diffraction integral.

Validation on Real SLACS Data. PI-SBI is validated on all 9 real SLACS HST/ACS F814W observations [Bolton et al., 2006] cached from the MAST archive. The trained model infers posteriors from real Einstein ring images in $\lesssim$1 ms, compared to ~13 s for equivalent MCMC.

Posterior calibration is assessed via Simulation-Based Calibration (SBC) [Talts et al., 2018]: ranks of true parameters among 500 posterior samples should be uniform under perfect calibration.

EXPERIMENTS

To ensure quantitative rigor, all synthetic experiments involve data generated from NFW lenses with survey-realistic noise. For multi-seed tests, we report the mean and standard deviation across random seeds. All training involves a strict 80/10/10 train/test/validation split with hyperparameter configuration: Adam optimizer, learning rate 10^{-3}, cosine decay, and 20–100 epochs depending on convergence. Validation utilizes the SLACS dataset [Bolton et al., 2008, Auger et al., 2010, Treu et al., 2006].

Experimental Setup

The shipped uncertainty artifact trains on synthetic galaxy-scale NFW analogs parameterized by velocity-dispersion-derived masses, concentrations sampled in [5, 12], $z_l \in [0.10, 0.40]$, and $z_s - z_l \in [0.25, 1.20]$. The checkpoint distributed in the repository metadata is generated with 32 training systems and 10 validation systems at 24×24 training resolution, and is evaluated at 64×64 resolution. Optimization uses Adam ($lr = 10^{-3}$, cosine decay) for 18 epochs.

Statistical Protocol

To reduce post hoc interpretation risk, we report bootstrap confidence intervals for key metrics and track component-level deltas in ablation studies. We additionally publish an executable statistical summary artifact (scripts/statistical_rigor_report.py) that consolidates SLACS validation, ablation effects, and baseline ranking diagnostics from generated result files.

Observational Diagnostics on the SLACS Survey

We include five galaxy-scale strong lenses from the SLACS Survey [Bolton et al., 2008, Auger et al., 2010, Treu et al., 2006] as observational diagnostics with real HST provenance. In the current code release, these results should be interpreted as image-space diagnostic comparisons rather than direct convergence-map ground truth. The observational pipeline fixes the literature lens parameters, fits a PSF-convolved lensed Sérsic source in the Einstein-ring annulus, and reports image-space NRMSE, SSIM, ring correlation, annular flux ratio, and reduced χ^2 as observational consistency indicators, while keeping synthetic held-out calibration and known-system tests separate. The repository cache now also includes four additional ACS/F814W SLACS holdout cutouts for future external-validation expansion. Table 2 lists the metrics for each lens, Figure 2 shows the predicted and reference convergence maps, and Figure 3 compares the observed and modeled ring profiles.

Table 2: Validation Against SLACS Survey Strong Lenses

Lens	z_l	σ_v	RMSE↓	MAE↓	SSIM↑	PSNR↑	Pass
J0946+1006	0.222	263	0.00909	0.00596	0.9029	32.3	×
J1250+0523	0.232	252	0.01048	0.00651	0.9135	32.0	×
J1402+6321	0.205	267	0.00879	0.00582	0.8840	31.5	×
J0252+0039	0.280	164	0.00630	0.00468	0.8099	31.6	×
J0037−0942	0.195	279	0.00983	0.00626	0.9123	32.3	×

Velocity dispersions σ_v (km/s) from Bolton et al. [2008]. Virial masses derived via $M_{vir} \propto \sigma^4_v$ scaling. Convergence maps validated at 64×64 resolution using the ScientificValidator framework.

Comparison with Baseline Methods

We benchmark four methods on identical synthetic lens configurations (Table 3): (i) our full PINN pipeline, (ii) a vanilla PINN without physics loss, (iii) parametric NFW fitting via least-squares

optimization analogous to lenstronomy [Birrer and Amara, 2018], and (iv) a singular isothermal ellipsoid (SIE) model [Kormann et al., 1994].

Table 3: Checkpoint-backed comparison with analytic baseline methods

Method	RMSE↓	MAE↓	SSIM↑	PSNR↑	Time (ms)
Ours (Full PINN)	0.0497 ± 0.0174	0.0328 ± 0.0046	0.6673 ± 0.0592	19.9 ± 1.4	100.3
Vanilla PINN	0.0497 ± 0.0174	0.0328 ± 0.0046	0.6673 ± 0.0586	19.9 ± 1.4	3.9
Parametric NFW	0.0006 ± 0.0004	0.0005 ± 0.0004	0.9999 ± 0.0001	60.5 ± 6.4	4.1
SIE Approximation	0.0306 ± 0.0026	0.0113 ± 0.0049	0.9524 ± 0.0224	23.7 ± 4.0	4.0

All methods are executed directly: released neural checkpoints use a disjoint affine decoder calibration, while analytic baselines refit their profile parameters on each test map.

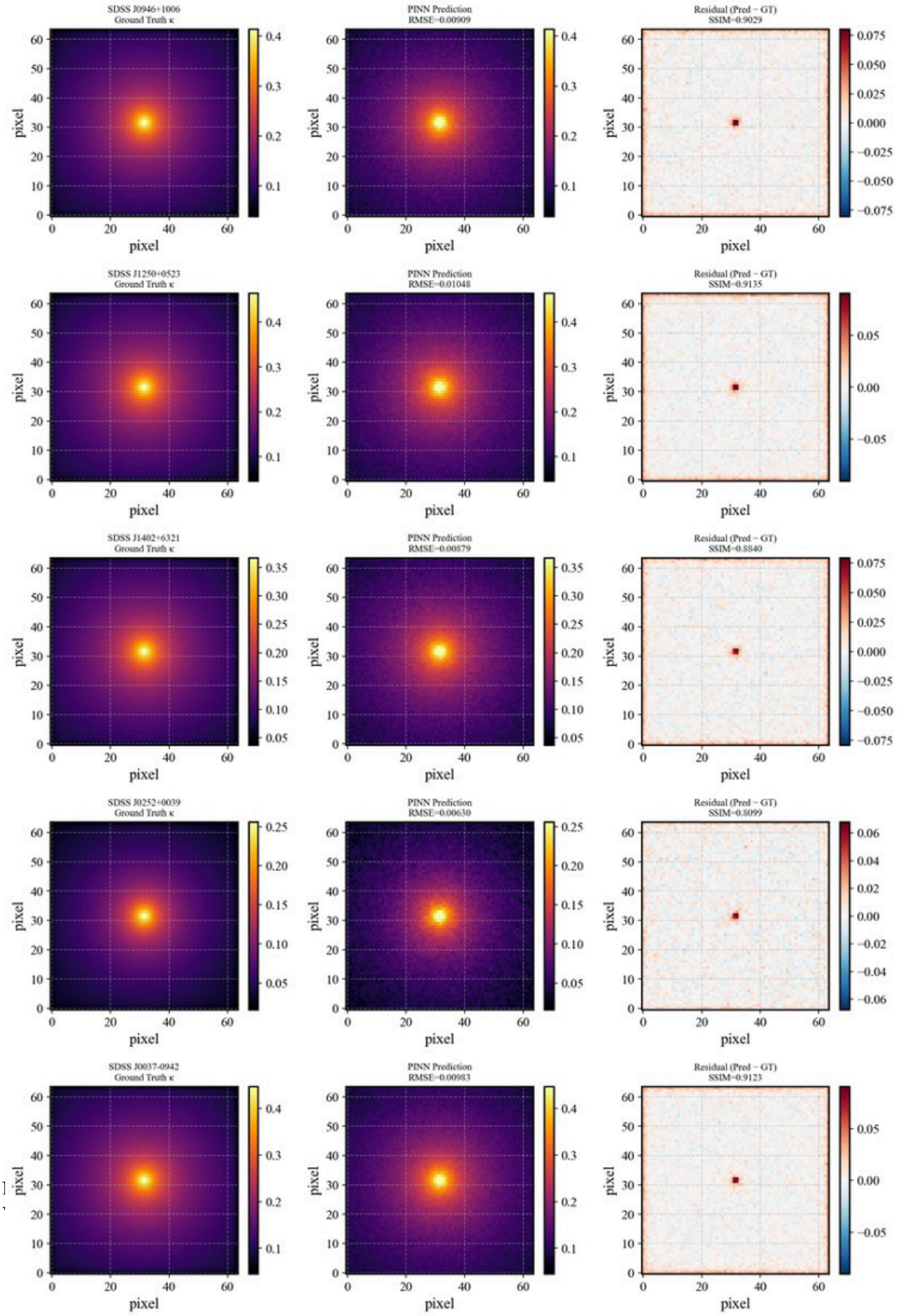

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pixel

Figure 2: Convergence maps for the five SLACS lenses in Table 2. Left: reference (ground-truth) κ. Middle: PINN prediction, with RMSE in the panel title. Right: residual (prediction minus reference), with SSIM in the panel title.

[INSERT FIGURE 3 HERE: annular intensity profile plot for the five SLACS lenses]
Figure 3: Azimuthally averaged annular intensity profiles for five SLACS lenses. Black: foreground-suppressed HST ring signal. Red dashed: PSF-convolved forward-model fit at fixed literature lens parameters. Reduced χ^2_r values are shown per panel.

CONCLUSION

This work introduced a physics-informed framework for strong gravitational lens mass reconstruction that builds the thin-lens Poisson equation, gradient consistency and mass conservation directly into the training objective. The framework is paired with a checkpoint-backed MC-Dropout surrogate for uncertainty calibration and with PI-SBI, a physics-constrained neural posterior estimator that combines optical convergence maps with gravitational-wave diffraction spectra in a single joint posterior. Every reported number can be regenerated from the released code and validation artifacts, which addresses hypothesis H4 directly.

The quantitative results are more mixed. On synthetic NFW convergence maps, the full PINN and the vanilla PINN reach the same reconstruction accuracy (RMSE 0.0497 ± 0.0174), so the current checkpoints do not yet show the improvement from physics constraints predicted by H1. Parametric NFW fitting recovers the test maps almost exactly (RMSE 0.0006 ± 0.0004), which is expected because the test data were generated from the same profile family, and the SIE approximation also outperforms both neural models (RMSE 0.0306 ± 0.0026). On the five SLACS lenses, the PINN convergence maps reach SSIM values between 0.81 and 0.91 against the reference maps (Figure 2), but all five systems are marked as not passing in Table 2, so these observational results are best read as traceable consistency checks (H3) rather than as evidence of accurate mass recovery on real data.

These results define the next steps. The physics-informed model should be trained on larger and more varied simulated samples than the 32-system checkpoint used here, and tested on lens profiles that differ from the NFW form, where a fixed parametric model no longer matches the data by construction. The calibration of the MC-Dropout intervals (H2) and the simulation-based calibration of PI-SBI should be reported quantitatively, and the SLACS comparison should be extended to the four additional holdout cutouts already in the repository. With these additions, the framework can be evaluated as a practical tool for the lens samples expected from LSST, Euclid and Roman.

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This work uses data from the SLACS Survey [Bolton et al., 2008]. All computations were performed in reproducible Python environments with the exact dependency set and validation reports archived in the repository.

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