

How Do the Axioms of Choice and Determinacy Create Notable Consequences Across Multiple Areas of Mathematics and to What Extent Is the Relatively Dominant Role of Choice Justified by These Consequences?

Murtuza Munir Farooqi
mmf7992@gmail.com

ABSTRACT

This paper provides an evaluation of the incompatibility of the Axioms of Choice and Determinacy within the Zermelo-Fraenkel set theory. This paper explores this discrepancy via observation of Gale-Stewart games on the Baire Space. By analyzing the effects caused by this discrepancy on multiple major fields of mathematics, a clear contrast between the strictness of the results of Determinacy and the pathological anomalies as a byproduct of the more permissible nature of Choice is observed. Ultimately, it is argued that the dominance of Choice over Determinacy, as seen throughout the more mainstream fields of Set Theory and Linear Algebra, is merely the result of the need for contextual consideration of each axiom's utility in each relevant field.

INTRODUCTION

Axioms are the foundation of all of mathematics and the leading, most widely accepted axiomatic framework is the Zermelo-Fraenkel Set theory (ZF). It contains eight central axioms, however in some frameworks it notoriously contains a controversial, but popular, ninth axiom. This axiom is known as the axiom of choice (AC) and it has been the subject of historic debate. There exist alternative axioms, such as the axiom of determinacy (AD) which are consistent with Zermelo-Fraenkel, however not with AC, and they can therefore be used as a replacement to AC.

How Do the Axioms of Choice and Determinacy Create Notable Consequences Across Multiple Areas of Mathematics and to What Extent Is the Relatively Dominant Role of Choice Justified by These Consequences?

This paper aims to understand the main implications of AC and AD across various fields of mathematics, and evaluate the consequences of their acceptance in each field. By the end, we will form a conclusion with our understanding of these aforementioned results to evaluate their role in modern mathematics and to formulate an answer for our research question. The importance of this research question stems from the implications of acceptance of either axiom to not only abstract fields like set theory, topology etc, but also to our intuitive understanding of the universe, as will be demonstrated. Additionally, quantum mechanics, one of mankind's most sophisticated models of the universe, depends on results that require AC (Brunner et al., 1996), highlighting its real-world significance. Finally, for most of these fields, there exists no singular answer or paper which defines the consequences of each axiom on each field, therefore it is important to help illustrate this, so this question does hold real world significance.

For the structure of this paper, we will begin with proper definitions of choice and determinacy and what they entail. We will then address the reason AC and AD are incompatible, which apart from being useful to understanding why exactly our future arguments hold weight, will also consider results which will be crucial in future sections. For our consequences, we will delve into four main fields, those being: 1) Set Theory and Measure Theory, 2) Algebra and Functional Equations, 3) Linear Algebra, 4) Topology and Functional Analysis. We will evaluate the effects of the acceptance of either axiom on each field and draw subconclusions to help us answer our question. Our evaluation of the 'how' part of the question will be conducted by constructing proofs through the consultation and extrapolation of mathematical literature in our main research body and 'to what extent' will be answered in our conclusion.

PRELIMINARIES AND DEFINITIONS

AC presents a way to form a new set from a family of non-empty sets by picking exactly one element from each set in the collection. Mathematically, AC states that it is possible for a choice function to be defined for a family of non-empty sets, it is possible to select an element for each set. AC asserts that there is at least one choice function on. Formally:

$$\forall S \left[(\forall X \in S (X \neq \emptyset)) \Rightarrow \exists \gamma : S \rightarrow \bigcup S \text{ such that } \forall X \in S, \gamma(X) \in X \right].$$

For a finite, a choice function can be defined by explicit construction, but there is no rule for an infinite. (Jech, 2003)

AD states that every infinite two player game of perfect information on the natural numbers is determined. Specifically, the infinite two-player game in which two players alternately choose natural numbers to form a sequence always has a winning strategy for one player. (Kechris, 1984)

THE MUTUAL EXCLUSIVITY OF AC AND AD

The Baire space, $\mathbb{N}^{\mathbb{N}}$ is the set which consists of all infinite sequences of natural numbers. It is the set of all functions from $\mathbb{N}$ to $\mathbb{N}$, $f : \mathbb{N} \rightarrow \mathbb{N}$. Fig. 3.1 illustrates how the Baire space can be thought of as a tree, where each branch is a separate sequence of natural numbers. Note that the Baire space is topologically equivalent to the irrationals, $\mathbb{R} \setminus \mathbb{Q}$ (Moschovakis, 2009)

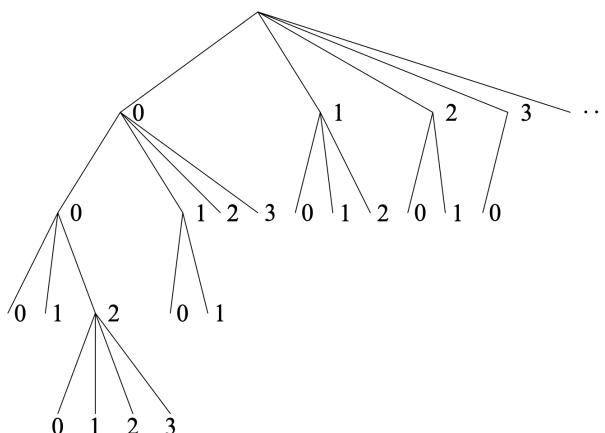


Fig 3.1. Picturing $\mathbb{N}^{\mathbb{N}}$ as a tree (Moschovakis, 2009)

The infinite game of perfect information (ie, a game in which the players' moves are visible to each other) G_X is played on X , a chosen subset of the Baire Space, which can be interpreted as a group of branches of the displayed tree. Since the Baire Space is topologically equivalent to the irrationals ($\mathbb{R} \setminus \mathbb{Q}$), X can be interpreted as a set of points on the real line. Two players, α_1 and α_2 play an infinite game, choosing alternate natural numbers, $n_0, n_1, n_2, \dots$ and so on. This game is infinite, meaning there is no terminal move. The sequence of chosen numbers converges to create a real number x , given by:

$$x = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}}$$

The outcome of the game is determined by the final value of x . α_1 wins in the case $x \in X$ and conversely α_2 wins in the case $x \notin X$. Now, we can introduce 'strategies' as mathematical functions which make a move based on the history of moves. Let σ represent the strategy of α_1 and τ represent the strategy of α_2 . σ and τ then output a natural number based on the previous moves.

How Do the Axioms of Choice and Determinacy Create Notable Consequences Across Multiple Areas of Mathematics and to What Extent Is the Relatively Dominant Role of Choice Justified by These Consequences?

$$\sigma : \bigcup_{n \in \mathbb{N}} \mathbb{N}^{2n} \rightarrow \mathbb{N} \qquad \tau : \bigcup_{n \in \mathbb{N}} \mathbb{N}^{2n+1} \rightarrow \mathbb{N}$$

AD postulates that there must always be a winning strategy for one player for any possible X (Mycielski and Steinhaus, 1962).

Keeping in mind the result given by AD, we will now examine a contradiction presented by AC. Khomskii (2010) considers G_X and uses transfinite induction, which extends basic induction from the natural numbers to the ordinals (which in our case is necessary since G_X is an infinite game). The collection of all possible strategies for α_1 and α_2 is uncountable, therefore AC can be applied to these collections to sort them into sequential lists indexed by ordinals, $\{\sigma_\beta\}$ and $\{\tau_\beta\}$ for both players respectively. For clarification, the ordinal β is the index used for the β -th strategy

Now, sets A and B will be constructed for G to be played on. At each stage of the transfinite induction, an outcome of the game, b_β , is considered which is consistent with α_1 's current strategy of σ_β and this outcome will be placed into B . Simultaneously, an outcome consistent with τ_β , α_2 's strategy, (which will be denoted by a_β) will be placed into A . Through this, we can conclude that for G_A , the game will be undetermined, since for any σ , there exists a b which will lie outside the set A and for any τ , there exists an a which will lie inside the set A . Since there is a valid outcome for τ that lies within A , and for α_2 to win, $a \notin A$, τ is not guaranteed to win. Similarly, since there is a valid outcome for σ that lies within B , and for α_1 to win, $b \in A$, σ is not guaranteed to win.

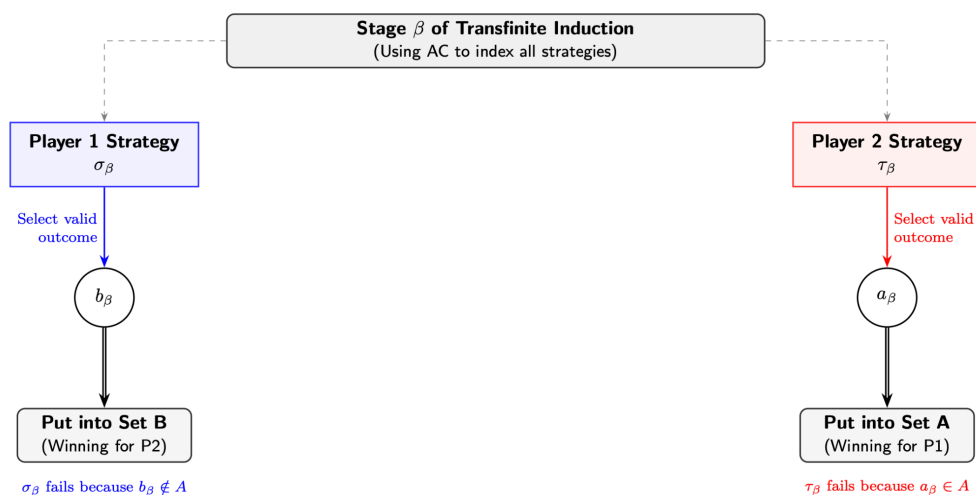


Fig 3.2. A visualization of transfinite induction (Original)

Therefore, this result directly contradicts the implication of G_X always being determined, which is the basis of AD. Thus, we can conclude that AC and AD are antithetical to each other, presenting a problem for modern mathematicians. Considering both axioms are indeed valid and internally consistent, which one should they use?

Beyond their mechanisms, a further concern arises from the consideration of the relative consistency strengths of $ZF + AC$ (ZFC) and $ZF + AD$. Gödel (1940) presented, in a landmark paper, that if the standard Zermelo-Fraenkel theory of sets was to be considered consistent, then adding AC to the axioms of ZF would not present contradictory results. By considering the constructible universe L , it has been shown that L is an inner model of ZF , every set in L is constructible by definition and every set can be well-ordered in L , therefore $L \models AC$. Since the model on which L has been constructed has been taken to consider ZF to be consistent within it, this implies that $L \models ZFC$, so every model of ZF contains an inner model satisfying ZFC and so $Con(ZF) \implies Con(ZFC)$. Since it is trivial to note that $Con(ZFC) \implies Con(ZF)$, ZF and ZFC are equiconsistent. Woodin (1999) establishes that $AD^{L(\mathbb{R})}$ (where $L(\mathbb{R})$ denotes the constructible universe over the reals) is equiconsistent with the assumption that infinitely many Woodin cardinals exist plus a measurable cardinal above it. Since this assumption extends beyond ZF, and $Con(ZF) \iff Con(ZFC)$, we can conclude that $Con(ZF + AD^{L(\mathbb{R})}) \implies Con(ZFC)$ and since AD is strictly stronger than $AD^{L(\mathbb{R})}$, $Con(ZF + AD) \implies Con(ZFC)$. The requirement of assumptions beyond ZF for $ZF + AD$ but not for ZFC leads to the consistency strength of ZFC being strictly lower than that of $ZF + AD$.

CONSEQUENCES IN SET THEORY AND MEASURE THEORY

We now demonstrate an example of the cost of the acceptance of AC and a fix provided by AD. Lebesgue measure is a property which allows a ‘measure’ to be assigned to a set, which can be thought of as the generalization of a property such as length, area and volume to any n -dimensional set. We now present a simplified sketch of the Mycielski-Świerczkowski (1964) argument that AD implies all sets of reals to be Lebesgue Measurable. Recalling the game defined previously in Section 3, we now propose an additional restriction. For each interval I_n , recall that $I_n \supseteq I_{n+1} \supseteq I_{n+2} \dots$, so I converges to some point x . Let $m(X)$ denote the Lebesgue measure of some set X . Our restriction states that $k_1 m(I_n) \leq m(I_{n+1}) \leq k_2 m(I_n)$ for fixed $0 < k_1 \leq k_2 < 1$. This bounds the rate of decay of the interval length, which allows the set of outcomes reachable under a given winning strategy to be bounded below via standard covering estimates. Since the game is determined, either player has a winning strategy. If α_1 has a winning strategy, then since $x \in X$, the set of all outcomes under the strategy of α_1 is a closed subset of X , and from our previous argument, has positive measure. If α_2 has a winning

strategy, then the outcomes of the strategy lie outside of X , and therefore we know $\mathbb{R} \setminus X$ contains a closed set of positive measure. Running this construction along a sequence of games on smaller

intervals squeezes the inner and outer measure of X together, establishing that X , and therefore any subset of the reals is Lebesgue measurable under AD.

This, however, is not the case for AC. Formally speaking, G_A is a Gale-Stewart game, and the condition causing Banach-Mazur games to differ from Gale-Stewart games is the selection of a non-empty open subset from the target set instead of a singular element (in our context, a natural number). However, by treating the sequence of elements as they are being selected like a shrinking interval, we can treat a Gale-Stewart game as a Banach-Mazur one, and therefore treat these games equivalently. We now observe the findings of Atchley et al. (2023). The Bernstein set is a set which itself, along with its complement, intersects every uncountable closed subset of $\mathbb{R}$ but contains none of these subsets (Oxtoby, 1980). We notice that A has specifically been constructed such that it is a Bernstein set since each strategy's outcome tree forms an uncountable closed set and A contains at least one (but not every) outcome, while the complement set similarly contains at least one outcome from the opposing strategy, thus fulfilling the requirements for being a Bernstein set. For a Banach-Mazur game and so, for G_A , it has been shown that for a target Bernstein set, the game is undetermined, verifying the incompatibility of AC and AD, and showing an example of the construction of a non-Lebesgue measurable set by AC.

There have been multiple notable constructions of other such non-measurable sets with the use of AC, like the Vitali set. By Addleman (2015), it is possible to construct a set V by choosing an element from each equivalence class in the interval $[0, 1]$, where each equivalence class is a set disjoint to all other equivalence classes containing all numbers differing by a rational number. Its non-measurability is a proven result. Given that V is disjoint with every identical set shifted by some non-zero rational number r within $[-1, 1]$, and the shifted set is a subset of the interval $[-1, 2]$ since all values within $V + r$ can now lie that range at a maximum. By the property of monotonicity (Royden and Fitzpatrick, 2010), we know that for $A \subseteq B$, $m(A) \leq m(B)$. Since $[0, 1]$ is a subset of the union of all possible $V + r$ (ie, the set formed by combining all possible $V + r$) and by the principle of countable additivity (Royden and Fitzpatrick, 2010), we know that the measure of the union of all disjoint subsets is equal to the sum of the measure of all the disjoint subsets, this implies $m([0, 1]) \leq \sum m(V + r)$, where $V + r$ is summated for all rational values of r , since it is obvious that the measure of all sets undergoing the shift will be greater than 1. Similarly, $\bigcup V + r \subseteq [-1, 2] \implies \sum m(V + r) \leq m([-1, 2])$. At this point, note that since the measure of any translated set is strictly invariant to the original measure, we can say that: $m([0, 1]) \leq \sum m(V) \leq m([-1, 2])$

Note that since $m([0, 1]) = 1$ and $m([-1, 2]) = 3$, the Lebesgue measure of the sum of the Lebesgue measure of V across all possible r must lie within $[1, 3]$. However, note that if $m(V) = 0$, $\sum m(V) = 0$, else if $m(V) > 0$, $\sum m(V) = \infty$ which brings about a contradiction, since we have already established that the measure should lie within 1 and 3. Hence, due to this contradiction, we

say V is a non-measurable set. This in turn also breaks the aforementioned principle of countable additivity, which leads to the intuitive explanation of one-dimensional length being broken down since the Lebesgue measure on a one-dimensional real space is no longer guaranteed to be the sum of the individual lengths of the space divided into disjoint segments.

A similar observation is made when another example is tested, however this time regarding three-dimensional space and objects. The infamous Banach-Tarski paradox states that in a Euclidean space of at least three dimensions, two arbitrary sets, bounded and containing some interior points, are equivalent by finite decomposition (Banach and Tarski, 1924). If we consider one of the arbitrary sets to be a sphere of a set radius, and the other set to be two spheres identical to that in the first set, this does imply that both structures are equivalent. This is possible by extending Hausdorff's paradox, which states that for an infinitesimally thin surface of a hollow sphere, it is possible to remove a countably infinite amount of points from the surface and be left with three identical sets of points, A , B and C . A known result is that $A \cong B \cup C$, meaning that A is functionally equivalent (congruent) to two copies of itself. This can be extended by imagining the surface of the sphere as growing thicker until the shape is a sphere completely filled, barring its centre. This shape is equivalent by finite decomposition to the solid sphere, therefore Hausdorff's technique can be applied to a solid sphere to show its equivalence to two identical copies of the original sphere (French, 1988).

Hausdorff's method uses a free group of rank 2 which is generated by 2 elements, σ and τ . Each variable represents a combination of transcendental angles of rotation (any combination cannot be undone to get back to the original state unless the inverse of that combination itself is applied directly). The possible rotations which can be applied can be separated into four main distinct branches, each branch starting with either σ , σ^{-1} , τ or τ^{-1} and then continuing onward with these four operations (Tomkowicz and Wagon, 2016. See Fig. 4.1). As per French (1988), we consider the set of all possible combinations of these transformations (rotations, to be specific) to be denoted by Q . Every non-trivial rotation in Q has an axis of rotation which intersects the sphere at exactly two points (which remain invariant upon that rotation being applied to the sphere). We construct a new set D which contains all of these points. Since D is infinitesimally small, since it is countably infinite, removing it from the sphere does not affect the total volume of the sphere. For this sphere, with all points contained within D removed, we can divide it into three congruent sets of points A , B and C by considering a family of sets $\chi(p)$. $\chi(p)$ contains any point on the sphere, p , and all possible combinations of rotations in Q applied onto it. $\chi(p) = \{p, \sigma(p), \tau(p), \sigma^{-1}(p), \tau^{-1}(p), \sigma\tau(p) \dots\}$. For each $\chi(p)$, we can use AC to pick a

single element and collect all these into a new set. On this set, three congruent pairs of transformations in Q can be applied to form sets A , B and C . We know that the original sphere S can be divided into four disjoint sets, A , B , C , and D , of which we previously have established that $A \cong B \cup C$. Because of this result, we know that A can be divided into two sets, A_1 and A_2 which are respectively congruent to B and C . Since we know that $A \cong B \cong C$, we can divide B and C similarly into B_1 , B_2 , C_1 and C_2 .

We know that:

$$\begin{aligned} S &= A \cup B \cup C \cup D \\ &= (A_1 \cup A_2) \cup (B_1 \cup B_2) \cup (C_1 \cup C_2) \cup D \\ &= (A_1 \cup B_1 \cup C_1 \cup D) \cup (A_2 \cup B_2 \cup C_2) \\ &= S \cup (S \setminus D) \end{aligned}$$

From the above proof, we can infer that since $S \setminus D$ has the same volume as S due to the infinitesimal size of D , then S has been decomposed into two spheres congruent to itself and has essentially doubled its volume. By considering AC to be true, such a proof allows for the disallowment of conservation of volume, leading to the breakdown of volume as we intuitively understand it. In this example, we have clearly seen how AC is more permissive in the context of allowing the construction of nonmeasurable sets, but also observe that AC leads to a pathological paradox, which contributes to the defense of AD.

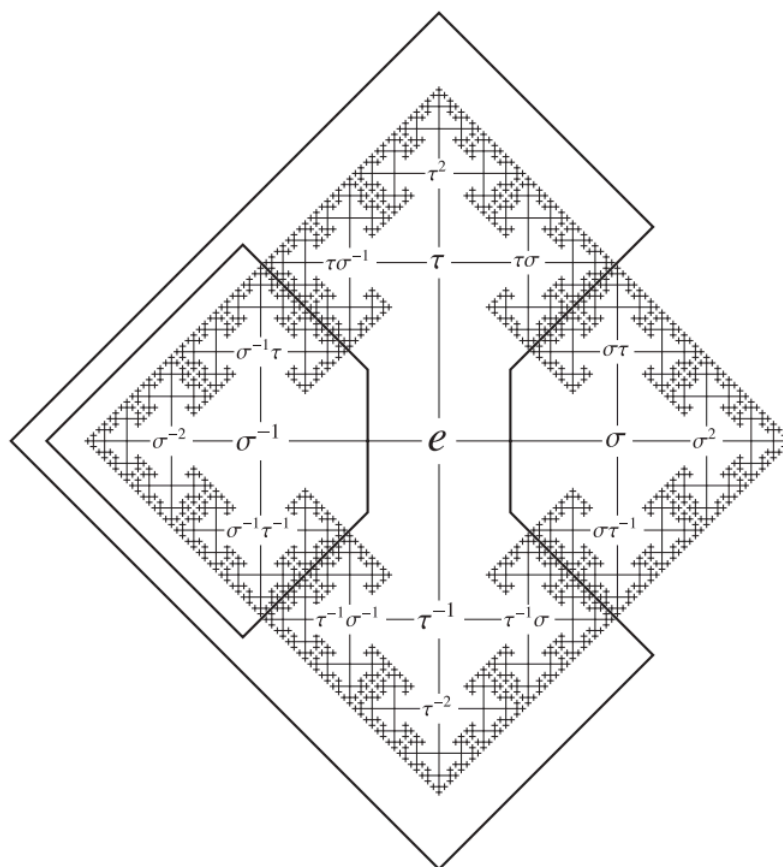


Fig 4.1. The free group (Tomkowicz and Wagon, 2016)

CONSEQUENCES IN ALGEBRA AND FUNCTIONAL EQUATIONS

We now consider the problem of functional equations and demonstrate how AD not only prevents paradoxical results produced by AC, but also provides a non-pathological result. By Aczél and Erdős (1965), the Cauchy additive functional equation is defined as:

$$f(x + y) = f(x) + f(y), f : \mathbb{R} \rightarrow \mathbb{R}$$

If we limit this relationship for the function to map the rationals onto the rationals, a simple induction proof has been given by Aczél and Erdős which nets $f(x) = \lambda x$ for the rationals.

By Sahoo and Kannappan (2010), this relationship can be extended onto all real x by considering a sequence of rational numbers converging to x , which will be denoted by $\{r_n\}$. Since for some natural n , we know that r_n is rational, we know that $f(r_n) = \lambda r_n$. By using our definition of $\{r_n\}$, we see that $f(x) = f(\lim_{n \rightarrow \infty} r_n)$. Because we assume f is continuous, $f(x) = \lim_{n \rightarrow \infty} f(r_n)$ by the definition of continuity. Our previous establishment makes noticing $f(x) = \lim_{n \rightarrow \infty} \lambda r_n$ an easy observation, which draws us to the conclusion that $f(x) = \lambda x$ for all real x , giving only one elementary solution.

We now consider to have no assumptions for f , ie. f can be allowed to be discontinuous. We consider the use of a Hamel basis for the next part which is defined as a set of vectors, a finite linear combination of which (accounting for scalar multiplication) spans the entire vector space and each element of the basis cannot be represented by a combination of the other vectors (Hamel, 1905). Following Jakobsen (2010) consider the Hamel basis of $\mathbb{R}$ as a vector space over $\mathbb{Q}$. This basis can be represented by $y_i \in \mathbb{R}$ where each y_i term is an element of the basis and $i \in I$, where I is an arbitrary finite set of natural numbers starting from 0, with each term being incremented by 1. For some $q_i \in \mathbb{Q}$, we can conclude that $x = q_1 y_1 + q_2 y_2 + \dots$ for finite y_i . If we let each $f(y_i) = \lambda_i$, $\lambda_i \in \mathbb{R}$, we know that $f(q_i y_i) = q_i \lambda_i$ by our first proof, therefore it can be concluded that:

$$f\left(\sum_{i \in I} q_i y_i\right) = \sum_{i \in I} q_i \lambda_i$$

It appears that this function would be linear since each $f(q_i y_i) = q_i \lambda_i$, however the linear relationship λ_i is broken if y_i does not remain constant for all i . Since each subsequent infinitesimally close value of x

will have separate linear combinations of y_i , each respective $\sum q_i \lambda_i$ will yield a wild, divergent output and therefore a pathological, scattered solution for $f(x)$ along the graph $G = \{(x, f(x)) : x \in \mathbb{R}\}$

Conversely, by Kuczma (2009), we know that every solution to Cauchy's additive function is convex, and by Sierpiński's theorem (which states that every measurable-convex function is continuous), we can conclude that the additive function would be continuous if it were measurable. Since we have concluded in Section 4 that by AD, any subset of $\mathbb{R}$ is measurable, f must indeed be continuous, therefore

disproving pathological solutions and leaving $f(x) = \lambda x$ as the only valid solution under AD. We have therefore demonstrated the example of a particular strong result only producible by AD.

CONSEQUENCES IN LINEAR ALGEBRA

Cherney, et al. (2013) defines the basis of a vector space as a set of linearly-independent vectors (ie, vectors which can never cancel out with non-zero coefficients) which also span the vector space. For example, for an n -dimensional real space, $\mathbb{R}^n$, the basis, S , is conventionally expressed as:

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

Observe that $\mathbb{R}^n = \text{span } S$ and all elements of S are linearly independent.

We now consider Zorn's Lemma. For a partially ordered set (a set paired with a binary relation $\preceq$ which relates some elements to others), there exist nonempty chains (totally ordered sets, with each element being related with the binary operation) which are subsets of the poset (partially ordered set). If each chain has an upper bound (element in the poset u such that each element in the chain $\preceq u$), Zorn's Lemma states that there exists at least one 'maximal' element in the poset m meaning there exists no element in the poset p , such that $m \prec p$ (Salamon, 2025). Friedberg et al. (2018) states that for a vector space V , the family of all linearly independent subsets of V that contain S , some linearly independent subset of V , is to be denoted by F . Consider a chain C in F and let $U = \cup_{C_r \in C} C_r$. U acts as an upper bound for C . Thus, $U \in F$ because any test for linear dependence strictly involves only a finite

number of vectors. Any finite selection of vectors from U must inevitably be contained entirely within a single, already-independent subset C_r from the chain, therefore U preserves linear independence. Since U is linearly independent and acts as an upper bound for C , Zorn's Lemma implies that F has a maximal element, which in turn implies that F always contains a maximal linearly independent subset of V (which cannot contain another vector without becoming linearly dependent).

Since any maximal linearly independent subset of V is equivalent to a basis of V , it is elementary to deduce that each vector space has a basis. Hence, Zorn's Lemma implies AC which implies the existence of a basis for any vector space. It is at this point, it becomes obvious that since we have referred to AC to guarantee the existence of a maximal element and have also explicitly shown that this leads to each vector space having a basis. Contrasting the previous sections, we have provided an example where we can observe a limitation within the acceptance of AD, since this result is nonreproducible in this context. We have observed an example where AC explicitly constructs a fundamental result.

CONSEQUENCES IN TOPOLOGY AND FUNCTIONAL ANALYSIS

We define, using Munkres (2000), an open covering to be a set of open subsets of a space such that the union of all elements in the set is equal to the space. Through this, we define compactness as a property of a space such that if every open cover of the space has a finite subset which also covers the space, the space is considered compact. Finally, the Finite Intersection Property (FIP) is the property of a collection of subsets of a set, such that any finite combination of these subsets has a non-empty intersection. Now we consider a collection of compact spaces, and their product, X :

$$X = \prod_{\alpha \in J} X_{\alpha}$$

The set A is a collection of subsets of X , such that it fulfills the FIP. For the poset $(F, \subseteq)$, where F is the set of all possible collections of subsets of X possessing the FIP, Zorn's Lemma implies the existence of D , a collection of subsets of X , such that $A \subseteq D$ and D is a maximal element of the poset. Since $D \in F$, D also has the FIP. Consider a projection map $\pi_{\alpha} : X \rightarrow X_{\alpha}$. Because every X_{α} is compact, the intersection of the closure of every $\pi_{\alpha}(D)$ is non-empty. x_{α} is arbitrarily chosen using AC such that each x_{α} is in the aforementioned closure. A unified point $x \in X$ is constructed as the tuple of

all x_{α} , $\alpha \in J$. Since x is contained within every set in D , x is also contained within every set in A . Because A is an arbitrary collection, it follows that for any arbitrary collection of sets with the FIP in X , there exists a non-empty intersection and so X will always be compact, which is the statement of Tychonoff's theorem: The product of compact spaces always yields a compact space. Therefore, AC

implies Tychonoff's theorem, and AD fails to do so since Zorn's Lemma, as we have established, does not imply AD. Furthermore, Kelley (1950) has exemplified that by considering a family of sets $\{X_a\}_{a \in A}$, another family of sets $\{Y_a\}_{a \in A}$ can be created such that $Y_a = X_a \cup \{p\}$ where $\{p\}$ is a single point. We define a topology on Y_a by considering the open sets to be the empty set and every subset of Y_a whose complement is finite, thus each Y_a is compact. By Tychonoff's theorem, we can say that, by definition, the cartesian product of all Y_a is compact. Let P be this product, and let $Z_a \subseteq P$, be the closed set of points where the a -th element lies in X_a . Using finite choice, every finite subcollection of Z_a has a non-empty intersection. Since P is compact, the FIP forces $\bigcap_{a \in A} Z_a \neq \emptyset$. Since any point in this full intersection is an element of $\prod X_a$, we can conclude that the cartesian product of any family of non-empty sets is non-empty, which is equivalent to AC, it has been proven that Tychonoff's Theorem implies AC. Since Tychonoff's theorem has been shown to be implied by AC at first, and then it has been shown to imply AC, we can say that it is logically equivalent to AC, and therefore we can conclude that AC is essential to the existence of Tychonoff's Theorem.

Though the existence of Tychonoff's Theorem does not prove much in a vacuum, it does yield the formation of the Banach-Alaoglu Theorem (Royden and Fitzpatrick, 2010) which is the basis of functional analysis. Through preserving compactness in infinite dimensional spaces, it is a popular way for modelling quantum mechanics and the acceptance of AC logically implies the validity of these models.

LIMITATIONS AND DIRECTIONS FOR FURTHER RESEARCH

The areas of discussion selected for consideration illustrate the most well-known and widely applicable examples in a wide range of mathematics. However, it would be uninformed to say that these are the only significant or relevant examples regarding the applicability of AC or AD paired with ZF Set Theory. Indeed, many examples in other branches of mathematics do exist which could provide useful insight into the discussion, however a lack of expository accessibility restricts the ability of discussion of these topics

within the scope of this paper. For the sake of brevity, we have elected to remain restricted to these fields, since elaborate setup and context of other specialized fields runs the risk of unneeded verbosity. Apart

from brevity, the fields have been chosen in a specific order for maximum clarity and interconnectedness. Set theory is the definitionally central field for AC, and Measure Theory was chosen, not only because the idea of measure was central to future arguments, but also because the Banach-Tarski Paradox remains to be among the most approachable, intuitive and understandable example in regards to the consequences of applying each axiom. Cauchy's Functional Equation provides an elegant statement, which might appear to be trivial but, in fact, is a great illustration of how, while seemingly invisible, the effect of the axioms is felt when the problem is observed through a more rigorous lens. Linear Algebra has been chosen because the existence of a basis for every vector space is a fundamental result in algebra, and is arguably the most significant argument based on applicability. Topology, with the example of Tychonoff's Theorem, deepens the argument, providing a concrete example where AC is unequivocally necessary.

One might find looking into pre-existing research insightful for drawing more solid conclusions. An example in combinatorics would be Hall's Marriage Theorem. While the finite case does not require AC, Hall's marriage problem requires use of Boolean Prime Ideal Theorem (Halbeisen, 2017), which is an implication of AC, and is also incompatible with AD. Another example in combinatorics can be found within Ramsey Theory. Though the Infinite Ramsey Theorem is compatible with both AC and AD, AD allows all sets of reals to possess the Ramsey Property (Woodin, 1999), which eliminates pathological sets created by AC. Another significant field of study would be within Model Theory. Namely, the Compactness Theorem for first-order logic is also equivalent to the Boolean Prime Ideal Theorem, and therefore also relies on AC (Halbeisen, 2017).

Apart from breadth, for a more complete answer to our question, reverse mathematics could be employed as a measure for rigorous analysis to classify the dependence of each field on AC or AD. While judgement on a case-to-case basis by analysing the importance of the axiom to the application is sufficient, a theoretical analysis using reverse mathematics, would indeed be the more rigorous option and therefore provide a more complete understanding for justification, though this paper does not employ it due to its high level of specialization. Apart from this, we are limited by what is considered "useful", mainly by using applications to justify our arguments. A philosophical perspective might enrich the argument by introducing an additional metric for justification, since historically, this debate has indeed

been significant regarding the discussion surrounding AC since its inception, with many different, long-standing schools of thought, all having valid but differing perspectives. Due to an issue in scope, however, philosophy has not been part of our discussion, though it is strongly recommended for the reader to familiarize themselves with these schools of thought for a more enriched stance on the problem.

CONCLUSION

The relative justification of choice over determinacy being more popular depends entirely on which lens it is observed through. As an example, when working within the topic of functional equations, one might consider the justification to be weak, since if they were to accept AC, the permission of non-elementary solutions to Cauchy's additive functional equation could very likely be grave for future works (by having notable, accepted models break down) but in that context, AD would be a crucial fix. Similarly, on the topic of measure theory, AD is necessary to help preserve regularity regarding Lebesgue Measure since AC would yield highly unusual results, similarly unnecessarily complicating further research. Conversely, if some mathematician needed to use the assumption of every vector space having a basis, AC would provide them with a valid framework which would have this result hold, and a mathematician needing to use Tychonoff's theorem would consider AC to be crucial, since it is logically equivalent to the theorem, and thus the acceptance of AD would invalidate the general result, thus rendering the justification of AC to either be extremely helpful or crucial from their respective perspectives. Additionally, the consideration of consistency strength adds another dimension to the discussion. Although consistency strength is, itself, not a criterion of correctness, it does remain an indicator of the strength of the assumptions required to establish their consistency. Whether greater consistency strength should be regarded positively or not is a question that would require specific context of the situation in which the axioms are applied.

Conclusively, the relative justification of AC over AD cannot be determined universally. The justification for each axiom's acceptance is contextual. The findings of this paper reveal that the justification of AC is relatively strong within the context of bases of vector spaces, by providing a sufficiently strong base framework to prove they always exist, and strong within topology, since Tychonoff's Theorem is logically equivalent to AC, and would not hold in full generality without it. Conversely, it is relatively weak within the context of measure theory and functional equations, hence the justification of the relative dominance of AC over AD is purely circumstantial.

REFERENCES

- Aczél, J. and Erdős, P. (1965) 'The non-existence of a Hamel-basis and the general solution of Cauchy's functional equation for non-negative numbers', *Publicationes Mathematicae Debrecen*, 12, pp. 259–263. <https://doi.org/10.5486/pmd.1965.12.1-4.25>
- Addleman, N. (2015) Non-measurable sets on the real line [Student Published Paper]. University of New Mexico. <https://math.unm.edu/~crisp/courses/math402/spring15/vitaliNikhil.pdf> (Accessed: 3 October, 2025)
- Atchley, J., Fishman, L. and Ghatti, S. (2023) 'Intersection games and Bernstein sets', *Involve: A Journal of Mathematics*, 18(5), pp. 813–820. <https://doi.org/10.48550/arXiv.2310.03039>
- Banach, S. and Tarski, A. (1924) 'Sur la décomposition des ensembles de points en parties respectivement congruentes', *Fundamenta Mathematicae*, 6(1), pp. 244–277. <http://eudml.org/doc/214280> (Accessed: 11th August, 2025)
- Brunner, N., Svozil, K., and Baaz, M. (1996). "The Axiom of Choice in Quantum Theory". *Mathematical Logic Quarterly*, 42, pp. 319-340. <https://doi.org/10.1002/malq.19960420128>
- Cherney, D., Denton, T., Thomas, R. and Waldron, A. (2013) *Linear Algebra*. Davis: UC Davis Mathematics.
- French, R.M. (1988) 'The Banach-Tarski theorem', *The Mathematical Intelligencer*, 10(4), pp. 21–28. <https://doi.org/10.1007/BF03023740>
- Friedberg, S.H., Insel, A.J. and Spence, L.E. (2018) *Linear Algebra*. 5th ed. Pearson.
- Gödel, K. (1940) *The Consistency of the Continuum Hypothesis*. Princeton University Press
- Halbeisen, L.J. (2017) *Combinatorial Set Theory with a Gentle Introduction to Forcing*. Springer
- Hamel, G. (1905) 'Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x+y)=f(x)+f(y)$ ', *Mathematische Annalen*, 60(3), pp. 459–462. <https://www.digizeitschriften.de/page/download> (Accessed: 25 October, 2025)

How Do the Axioms of Choice and Determinacy Create Notable Consequences Across Multiple Areas of Mathematics and to What Extent Is the Relatively Dominant Role of Choice Justified by These Consequences?

- Jakobsen, S.K. (2010) Cauchy's functional equation [Online Notes]. Available at: https://sunejakobsen.wordpress.com/wp-content/uploads/2010/12/cauchy_eng4.pdf (Accessed: 24 October, 2025)
- Jech, T. (2003) Set Theory. The Third Millennium Edition, revised and expanded. Springer.
- Kechris, A.S. (1984) 'The Axiom of Determinacy Implies Dependent Choices in $L(R)$ ', Journal of Symbolic Logic, 49(1), pp. 161–173. <https://doi.org/10.2307/2274099>
- Kelley, J.L. (1950) 'The Tychonoff theorem implies the axiom of choice', Fundamenta Mathematicae, 37(1), pp. 75–76. <http://eudml.org/doc/213229> (Accessed: 2 November, 2025)
- Khomskii, Y. (2010) Infinite Games and the Axiom of Determinacy [Online notes]. Universität Hamburg. <https://www.math.uni-hamburg.de/home/khomskii/infinitegames2010/Infinite%20Games%20Sofia.pdf> (Accessed: 20th September, 2025)
- Kuczma, M. (2009) An Introduction to the Theory of Functional Equations and Inequalities. Birkhäuser.
- Moschovakis, Y.N. (2009) Descriptive Set Theory. 2nd ed. American Mathematical Society.
- Munkres, J.R. (2000) Topology. 2nd ed. Prentice Hall.
- Mycielski, J. and Steinhaus, H. (1962) 'A mathematical axiom contradicting the axiom of choice', Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 10, pp. 1–3. <https://doi.org/10.2307/2271562>. (Accessed: 16th September, 2025)
- Mycielski, J. and Świerczkowski, S. (1964) 'On the Lebesgue measurability and the axiom of determinateness', Fundamenta Mathematicae, 54(1), pp. 67–71. <http://eudml.org/doc/213774> (Accessed: 16th September, 2025)
- Oxtoby, J.C. (1980) Measure and Category. 2nd ed. Springer-Verlag.
- Royden, H.L. and Fitzpatrick, P.M. (2010) Real Analysis. 4th ed. Prentice Hall.
- Sahoo, P.K. and Kannappan, P. (2010) Introduction to Functional Equations. CRC Press.
- Salamon, D.A. (2025) Notes on the Axiom of Choice [Online Notes]. ETH Zürich. <https://people.math.ethz.ch/~salamon/PREPRINTS/CHOICE.pdf>
- Tomkowicz, G. and Wagon, S. (2016) The Banach-Tarski Paradox. 2nd ed. Cambridge University Press.

How Do the Axioms of Choice and Determinacy Create Notable Consequences Across Multiple Areas of Mathematics and to What Extent Is the Relatively Dominant Role of Choice Justified by These Consequences?

Woodin, W.H. (1999) *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*. De Gruyter.